Parameterization and Controllability of Linear Time-Invariant Systems

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Abstract
This paper presents a generalized method to implement the idea of parameterization and model realization to synthesize a control function that affects a possible state transfer that are equivalent to the controllability of the Linear Time-Invariant (LTI) control systems.

The key idea here is that one should be able to link LTI control system with its system variables in parameterized form by the generalized term or the parametric function as mentioned in the generalized method, so that analysis of the system under study can be simplified. A generalized method is illustrated with significant results.
Keywords: *- parameterization, controllability, Linear Time-Invariant systems.*

1. Introduction

The theory of controllability plays an important role in design of modern control systems. The development in this field appears to be proceeding along several distinct lines but mostly in parallel manner [1-4]. In studying the significant aspect of these different approaches, the problem based on the basic concept of controllability is considered here.

In [4], concept of controllability is considered as a state transfer problem [STP] and proposed several methods for synthesizing a control function for steering the given initial state of the system to the origin. The solution to state transfer problem was based on relating the given system to a family of phase variable canonical form systems and then by using the technique of Hermits interpolation, it is possible to synthesize a control function.

Parameterization is the process of deciding and defining the parameters necessary for a complete or relevant specification of a model. Mostly it is a mathematical process of involving the identification of a complete set of effective coordinates of the system or model. The concept of parameterization of boundary-value control systems with pseudo-differential operator have been discussed in [5]. The key idea here is to parameterize the system variables by some external parameters, to design linear boundary-controlled partial differential equation (PDE) systems. Parameterization and approximation methods in feedback theory with application in high-gain, fast sampling and cheap-optimal control based on first-order plant models have been discussed in [6]. The idea of parameterization and its extension to study controllability of the linear system with method of Laplace transformation have been discussed in [7].

We have extended ideas of parameterization to study controllability of linear time-invariant systems without any Laplace transformation approach and synthesize a control function $u(t)$, which affects a possible state transfer of the system that satisfies conditions that are equivalent to the controllability of the system.

2. Methodology

Consider the time-invariant linear system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$

where $x(t) \in R^n$, $u \in R^m$, A, B are constant matrices. The equivalent representation in scalar differential equation of (1) as,

$$D^n x(t)+a_1 D^{n-1}x(t)+---+a_{n-1}Dx(t)+a_n x(t =u(t) \qquad (2)$$

where $a_1, a_2, ---, a_n$ are constant. The system (2) has a single input $u(t)$ and the system state can be described by the values of $x(t), Dx(t),----, D^{n-1}x(t)$ at each instant of time. Here we consider problem of controllability as a state transfer problem. If $x(t)$ is any n-times differentiable function such that it and its derivatives $Dx(t),--, D^{n-1}x(t)$ have the specified values at time t_i and t_f i.e.at initial and final time of specified time interval $[t_i, t_f]$, then expression (2) looks like a formula for $u(t)$ in terms of the response $x(t)$. It is possible to determine a control function $u(t)$ which affects possible state transfer, as providing two point boundary conditions on an n-times differentiable function $x(t)$. Such a system is said to be state controllable.
The system (2) can be parameterized as

$$D^n \varphi(t)+a_1 D^{n-1}\varphi(t)+--+a_{n-1}D\varphi(t)+a_n\varphi(t)=u(t) \qquad (3)$$

where $x(t) = \varphi(t)$ is any n-times differentiable function.
The system (1) can rewrite as

$$(DI_n - A)\, x(t) = Bu(t) \qquad (4)$$

where DI_n is a diagonal matrix, with all entries D on the diagonal. The solution $x(t)$ of (1) will be

$$x(t) = |DI\text{-}A|\ [DI_n\text{-}A]^{-1}.\ B\,u(t) \qquad (5)$$

As per binomial expression given in [8],

$$x(t)=|\,DI - A\,|\left[\left(\frac{I}{D}\right)+\left(\frac{1}{D^2}\right)A+\left(\frac{1}{D^3}\right)A^2+--\right]Bu(t) \qquad (6)$$

If $p(D) = |DI-A|$ is the characteristic polynomial of A, of degree n and $p_1(D)$, $p_2(D)$,----, $p_{n-1}(D)$, $p_n(D)$ are its associated polynomials as given below

$$p(D) = D^n + a_1 D^{n-1} + \text{-----} + a_{n-1}D + a_n ,$$
$$p_1(D) = D^{n-1} + a_1 D^{n-2} + \text{-----} + a_{n-2}D + a_{n-1} ,$$
$$p_2(D) = D^{n-2} + a_1 D^{n-3} + \text{-----} + a_{n-3}D + a_{n-2} ,$$

$$p_{n-1}(D) = D + a_1,$$
$$p_n(D) = 1 \qquad (7)$$

then solution $x(t)$ of (1) will be

$$x(t) = [p_1(D)\, I + p_2(D)\, A + p_3(D) A^2 + \text{--}]\, Bu(t)$$
$$= \varphi_1(t)\, B + \varphi_2(t)\, AB + \varphi_3(t)\, A^2 B + \text{--} + \varphi_{n-1}(t).\ A^{n-2}B + \varphi_n(t) A^{n-1}B \qquad (8)$$

where $\varphi_1(t) = p_1(D)\, u(t)$, etc. It is observed for (8) that solution $x(t)$ is spanned by the n-vectors

$$[B, AB, A^2B, \text{-----}, A^{n-1}B] \qquad (9)$$

If (4) is controllable, then it should be possible to transfer the system state from $x(0)$ to any $x(T)$ within specified time interval $[0, T]$. So this set (9) must be basis for R^n.

If $x(t)$ is a solution of (4) and (9) is the basis for R^n, then for each t, $x(t)$ is a linear combination of these vectors and so there exist functions $\varphi_1(t)$, $\varphi_2(t)$,----,$\varphi_n(t)$ such that (8) is true. Now substitute this expansion for $x(t)$ into (1), use the fact that the characteristic polynomial annihilates the matrix A.

$$A^n + a_1 A^{n-1} + \text{--} + a_{n-1}A + a_n I_n = 0 \qquad (10)$$

If we multiply by B and rearranged, we get

$$A^nB = - a_1 A^{n-1}B - a_2 A^{n-2} B - \text{----} - a_{n-1}AB - a_nB$$

so that A^nB is a linear combination of (9). Substitute (8) into (1) and simplify

$$\dot\varphi_1(t)B + \dot\varphi_2(t)AB + \text{---} + \dot\varphi_{n-1}(t)A^{n-2}B + \dot\varphi_n(t)A^{n-1}B$$
$$= \varphi_1(t)AB + \varphi_2(t)A^2B + \text{----} + \varphi_{n-1}(t)A^{n-1}B - a_1\varphi_n(t)A^{n-1}B - a_2\varphi_n(t)A^{n-2}B - \text{----} - a_{n-1}\varphi_n(t)AB - a_n\varphi_n(t)B + Bu(t) \qquad (11)$$

Match the coefficient on both the sides of these vectors to obtain the equations with, ‘ • ’ or ‘D’ represent as differential operator with respect to time t.

$$\dot\varphi_1(t) = - a_n\varphi_n(t) + u(t)$$
$$\dot\varphi_2(t) = - a_{n-1}\varphi_n(t) + \varphi_1(t)$$
$$\dots\dots\dots$$
$$\dot\varphi_{n-1}(t) = - a_2\,\varphi_n(t) + \varphi_{n-2}(t)$$
$$\dot\varphi_n(t) = - a_1\varphi_n(t) + \varphi_{n-1}(t) \qquad (12)$$

Thus it is possible to obtain the system (1) in the form of scalar differential equation with the parameterize function $\varphi_n(t)$ from (12). We can eliminate all these functions $\varphi_i(t)$, $i = 1, 2, \text{---}, n-1$ except $\varphi_n(t)$ by differentiating the second equation in (12) once, the third equation twice, ------ and the last equation $(n-1)$ times and by adding to get,

$$D^n\varphi_n(t) + a_1D^{n-1}\varphi_n(t) + a_2D^{n-2}\varphi_n(t) + \text{----} + a_{n-1}D\varphi_n(t) + a_n\varphi_n(t) = u(t) \qquad (13)$$

Thus the state transfer problem for (1) reduces to the state transfer problem for (13), which can be solved easily by interpolation. The response $x(t)$ determines $\varphi_1(t),\varphi_2(t)$,--,$\varphi_n(t)$ uniquely from (8) and $\varphi_1(t)$, $\varphi_2(t)$,-----,$\varphi_n(t)$ determines $D^{n-1}\varphi_n(t)$, $D^{n-2}\varphi_n(t)$,-----, $D\varphi_n(t)$ and $\varphi_n(t)$ from (12).

Further, instead of solving (13) for response $\varphi_n(t)$, we look it as a formula for a control function $u(t)$, which transfer the system states of linear time-invariant system and satisfies conditions that are equivalent to the controllability of the system.

3. Discussion of Results

Consider the system (1) as,

$$\dot x(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

with boundary conditions for $T = 1$ sec.

$$x(0) = [0 \quad 1 \quad -2]^T \text{ and } x(1) = [0 \quad 0 \quad 0]^T \qquad (14)$$

Here we are considering problem of controllability as the problem of state transfer. Hence we are synthesizing a control function $u(t)$ which transfer the system states from $x(0)$ to $x(T)$ within specified time interval of $T = 1$ sec. For the system (14) is to be controllable, solution $x(t)$ must be spanned by the controllability space formed by n-vectors $[B, AB, A^2B,\text{---},A^{n-1}B]$. The controllability subspace $\langle A|B\rangle$ is the space spanned by the column of matrix B with respect to the linear transformation A. This is necessary and sufficient condition for the pair $\langle A|B\rangle$ to be controllable [4]. If (14) is to be controllable, then it should be possible to transfer the system state from $x(0)$ to any $x(T)$ within specified time interval $[0,T]$. So the set (9) must be basis for R^n.

If $x(t)$ is a solution of (14) and (9) is the basis for R^n, then for each t, $x(t)$ is a linear combination of these vectors and so there exist functions $\varphi_1(t)$, $\varphi_2(t)$,----,$\varphi_n(t)$ such that (8) is true. Hence solution $x(t)$ of (14) as,

$$x(t) = \varphi_1(t)B + \varphi_2(t)AB + \varphi_3(t)A^2B \qquad (15)$$

By substituting (15) in (14) and with some efforts we can show that,

$$\dot\varphi_1(t)B + \dot\varphi_2(t)AB + \dot\varphi_3(t)A^2B$$
$$= (-6\varphi_3(t) + u(t))B + (-11\varphi_3(t) + \varphi_1(t))AB + (-6\varphi_3(t) + \varphi_2(t))A^2B$$

Match the coefficients of these vectors on both sides to obtain the equations as,

$$\dot\varphi_1(t) = - 6\varphi_3(t) + u(t)$$
$$\dot\varphi_2(t) = - 11\varphi_3(t) + \varphi_1(t)$$
$$\dot\varphi_3(t) = - 6\,\varphi_3(t) + \varphi_2(t) \qquad (16)$$

We can eliminate $\varphi_1(t)$, $\varphi_2(t)$ except $\varphi_3(t)$ by differentiating the second equation in (16) once, the third equation twice and by adding, we get the system (14) into an equivalent form with the parameterized function $\varphi_3(t)$ as,

$$D^3\varphi_3(t) + 6D^2\varphi_3(t) + 11D\varphi_3(t) + 6\varphi_3(t) = u(t) \qquad (17)$$

The terminal values for the parameterize function $\varphi_i(t)$ can be determined uniquely from (15) for boundary conditions as,

$$\varphi_3(0) = [0 \quad 1 \quad -2]^T, \ \varphi_3(1) = [0 \quad 0 \quad 0]^T$$

By using polynomial interpolation approach we get response $\varphi_3(t)$ for (17) with $(2n-1)^{th}$ degree polynomial and a control function $u(t)$ as,

$$\varphi_3(t) = t - t^2 - 3t^3 + 5t^4 - 2t^5 \quad (18)$$

$$u(t) = -19 - 4t + 135t^2 - 38t^3 - 80t^4 - 12t^5 \quad (19)$$

with terminal values $u(0) = -19$ and $u(1) = -18$.

Further, from (8), we get expression for the state variables for the system (14) as:

$$x_1(t) = \varphi_3(t) = t - t^2 - 3t^3 + 5t^4 - 2t^5 \quad (20)$$

$$x_2(t)=D\varphi_3(t)=1-2t-9t^2+20t^3-10t^4 \quad (21)$$

$$x_3(t)=D^2\varphi_3(t)= -2 -18t+60t^2-40t^3 \quad (22)$$

The transfer characteristic of a control function $u(t)$ and the system states $x_1(t)$, $x_2(t)$, $x_3(t)$ that satisfies boundary conditions are as shown in Fig. 1, 2, 3 & 4 respectively.

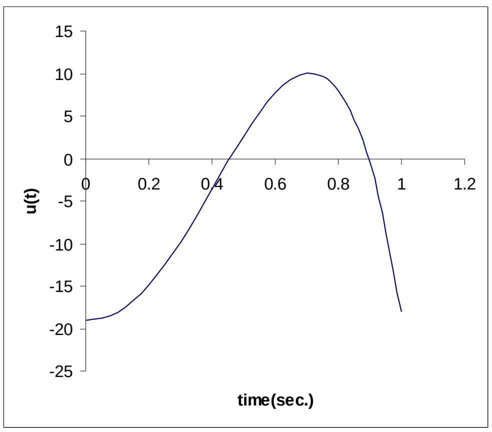

Fig.1 Transfer char. of a control function $u(t)$

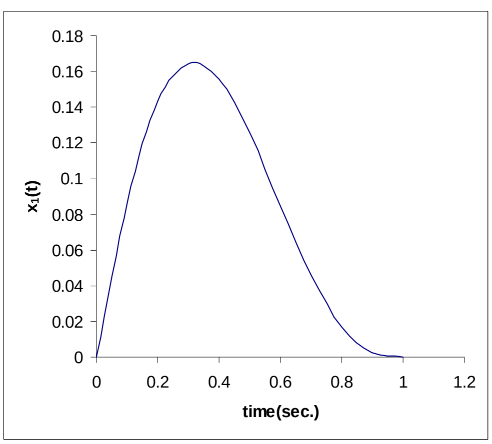

Fig. 2 Transfer char. of the system state $x_1(t)$.

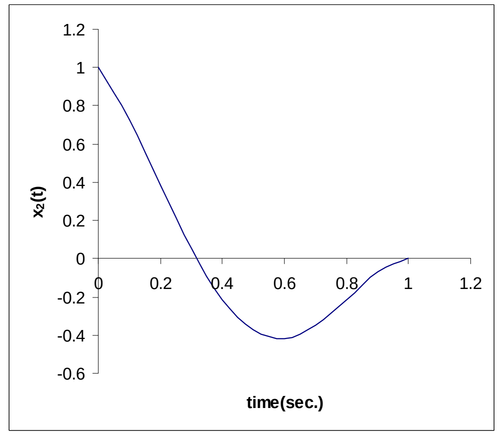

Fig. 3 Transfer char. of the system state $x_2(t)$.

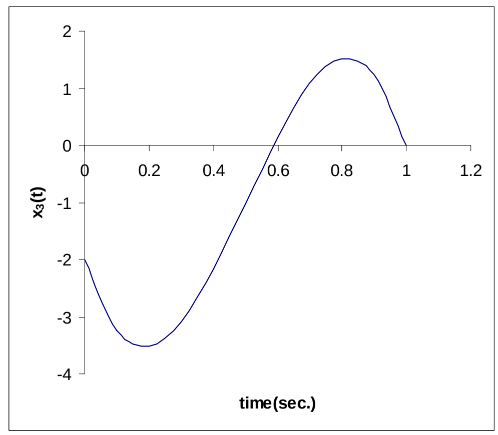

Fig.4 Transfer char. of the system state $x_3(t)$.

4. Conclusion

The main objective in design of modern control system is its controllability. This paper presents problem of controllability as a state transfer problem. Here a simple solution is proposed for the problem of determining a control function in parameterized form that affects a possible state transfer of a LTI system. The solution is based on relating the given system to a family of scalar differential equation in parameterized form and solving problem latter by two point interpolation.

A simple generalized method is presented here to analyze LTI systems in parameterized form. This generalized approach may extend to analyze any other parameterized LTI control systems so that study can be simplified. It is possible to realize the concept of controllability without using any laplace approach. This generalized approach has few more advantages as follows :

i) No computation of the state transition matrix is involved.

ii) No computation of the eigenvalues of system matrix A is required to check the suitability of the duration T of control.

iii) This method has the flexibility of choosing the time interval during which the transfers of the system states from initial to final values are desired.
iv) It applies to uncontrollable systems as well with suitable modifications. For uncontrollable systems, not all state transfers to the origin are possible.

This method of solving the state transfer problem can be generalized to those classes of functions for which the two-point interpolation problem can be solved. It may be possible to extend this approach for time-varying systems and singular systems.

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