On the local controllability of a discrete-time inhomogeneous multi-input bilinear systems

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Abstract

This paper studies the local controllability of a class of discrete-time inhomogeneous bilinear systems. A sufficient condition for the local controllability is proposed and the form of the optimal control is also presented. Furthermore, the established results are illustrated by an example and numerical simulation.

Keywords: Bilinear systems, discrete time, local controllability, optimal control.

1. Introduction

Bilinear systems are a special class of nonlinear systems; they form a transitional class between the linear and the general nonlinear systems. Through nearly half a century, they have received great attention by researchers. The importance of such systems lies in the fact that many important processes, not only in engineering [1], but also in biology [2], socio-economics [3], and chemistry [4-5], can be modeled by bilinear systems[6].

In the literature, several papers address the problem of controllability for bilinear systems. In [7], we raise two conditions for controllability: one for necessity and the other for sufficiency. Such approach is a local one and consists in decomposing the bilinear system model into a linear system and a multiplicative feedback. However, it requires that rank(Q) = 1 where Q must be factorized in two vectors; in other terms this technic needs orthogonality property. The same problem as considered in [8] gives rise to a global necessary and sufficient condition. In addition to decomposing the system as in [7], the approach involves forward and backward composition of the transition function. It still ensues a condition of orthogonality on the matrix Q, plus an inversibility condition on the matrix A. etc

The present paper deals with the question of local controllability for discrete time inhomogeneous multi-input bilinear systems. We adopt a method based on the linearization of the system and the definition of an appropriate operator that leads to the control transferring the system to a desired given state with a minimum energy.

The paper is organized as follows. In section 2, we present an approximation of the final state. Section 3 is aimed to the presentation of a sufficient condition for local controllability of an inhomogeneous multi-input bilinear discrete-time system. Section 4 provides an expression of an optimal control that can transfer the system from the initial state to a final desired state. Finally, an example of controllable bilinear systems is provided in section 5.

2. An approximation of the final state

In this article we consider the following inhomogeneous discrete-time bilinear system:

$$x(k+1) = Ax(k) + \sum_{i=1}^{p} u_i(k) B_i x(k) + Bu(k) \qquad (1)$$

where $x(k)$ is the n-dimensioned state vector at time k, $u(k) = (u_i(k))$ is the p-dimensioned control vector at time k, B is a matrix of dimension $n \times p$, A and $B_1;\ldots;B_p$ are square matrices of order n.

Let x_N denotes the final state and

$$x(k+1) = Ax(k) + \sum_{i=1}^{p} u_i(k) B_i x(k) + Bu(k) = F(x(k), u(k))$$

where F is a continuous vector function.

Let also $B = (b_{ij})$ with $b_{ij} \in \mathbb{R}$ for $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,p\}$, then the system (1) becomes

$$x(k+1) = Ax(k) + \sum_{i=1}^{p} u_i(k) B_i x(k) + V_{u_i(k)} = F(x(k), u(k))$$

where $V_{u_i(k)} = \left(\sum_{i=1}^{p} b_{1i} u_i(k) \ \cdots \ \sum_{i=1}^{p} b_{ni} u_i(k) \right)^T$

Consider the following function composition

$$x(N) = F_{u(N-1)} \circ \ldots \circ F_{u(1)} \circ F_{u(0)}(x(0)) \qquad (2)$$

with $F_{u(k)}(x(k)) = F(x(k),u(k))$

Using Taylor's development to expand the right-hand side of the previous equation yields:

$$x(N) = F_{u(N-1)} \circ \ldots \circ F_{u(1)} \circ F_{u(0)}(x(0))\big|_{\underline{u}=0} + \left[\frac{\partial F_{u(N-1)}(x(N-1))}{\partial u(N-1)} \ \frac{\partial F_{u(N-1)}(x(N-1))}{\partial u(N-2)} \ \ldots \frac{\partial F_{u(N-1)}(x(N-1))}{\partial u(0)}\right]_{\underline{u}=0} \underline{u}=0 + O\left(u^2\right)$$

with $\underline{u} = \left(u(N-1)\ldots u(1) \ \ u(0)\right)^T$, which can be rewritten as:

$$\bar{x}(N) = F_{u(N-1)} \circ \ldots \circ F_{u(1)} \circ F_{u(0)}(x(0))\big|_{\underline{u}=0} + P\big|_{\underline{u}=0} \ \underline{u} + O\left(u^2\right)$$

where

$$P = \left[P_{N-1} \ \mathrm{P}_{N\text{-}2} \ \ldots \mathrm{P}_0\right] = \left[\frac{\partial F_{u(N-1)}(x(N-1))}{\partial u(N-1)} \ \frac{\partial F_{u(N-1)}(x(N-1))}{\partial u(N-2)} \ \ldots \frac{\partial F_{u(N-1)}(x(N-1))}{\partial u(0)}\right]$$

From the equation (1) we have

$$\frac{\partial F_{u(k)}(x(k))}{\partial x(k)} = A + \sum_{i=1}^{p} u_i(k) B_i \ \text{ and}$$

$$\frac{\partial F_{u(k)}(x(k))}{\partial u_i(k)} = B_i x(k) + V_i \ \text{ with } V_i = \frac{\partial V_{u_i(k)}}{\partial u_i(k)} = \begin{pmatrix} b_{1i} \\ \vdots \\ b_{ni} \end{pmatrix}$$

for $i = 1,\ldots,p$

So by computing P after function composition, when controls are assumed to be equal to zero, we obtain

$$P = \begin{bmatrix} \left(B_1 A^{N-1} x(0) + V_1 \ldots B_p A^{N-1} x(0) + V_p\right) \\ \left(A\left(B_1 A^{N-1} x(0) + V_1\right) \ldots A\left(B_p A^{N-2} x(0) + V_p\right)\right) \\ \vdots \\ \left(A^{N-1}\left(B_1 A^{N-1} x(0) + V_1\right) \ldots A^{N-1}\left(B_p x(0) + V_p\right)\right) \end{bmatrix}^T$$

In other words, an approximation of the final state x_N when neglecting higher order control terms can be expressed as:

$$\bar{x}(N) = A^N x(0) + \sum_{k=0}^{N-1} \sum_{i=1}^{p} A^{N-1-k}\left(B_i A^k x(0) + V_i\right) u_i(k) \qquad (3)$$

Note that, in the rest of this work, we neglect higher order control terms. This assumption gives a local criterion of controllability.

3. A sufficient condition of local controllability

In this section we propose a sufficient condition of local controllability for the system (1).

First recall the definition of the local controllability for the systems (1).

Definition 1

The system (1) is said to be locally controllable on $I = \{0,1,\ldots,N-1\}$ for any x_0 and x_d from $\mathbb{R}^n$; there exists a control $u = \left(u_0, u_1, \ldots, u_{N-1}\right)$ as $\bar{x}_N = x_d$; where $\bar{x}_N$; given by (3), is the approximate solution of (1) at instant N corresponding to the initial state x_0 and the control u.

Then, let consider the operator defined by

$$H : \left(\mathbb{R}^p\right)^N \to \mathbb{R}^n$$

$$u = \left(u(0),\ldots,u(N-1)\right)^T \to Hu = \sum_{k=0}^{N-1} \sum_{i=1}^{p} A^{N-1-k}\left(B_i A^k x(0) + V_i\right) u_i(k) \qquad (4)$$

with $u(k) = \left(u_1(k) \ldots u_p(k)\right)^T \quad \forall u_i(k) \in \mathbb{R}$

Proposition 2

The operator H is linear, continuous and its adjoint operator H^* is given by:

$$H : \mathbb{R}^n \to \left(\mathbb{R}^p\right)^N$$

$$x \to H^* x = P^T x$$

Proof.

- The linearity of H is obvious
- For the continuity of H we show the existence of a constant $\alpha > 0$ such as

$$\|Hu\| \le \alpha \|u\| \quad \forall \mathrm{u} \in \mathrm{L}^2\left(0, N-1, \mathbb{R}^p\right)$$

$$\|Hu\| = \left\| \sum_{k=0}^{N-1} \sum_{i=1}^{p} A^{N-1-k}\left(B_i A^k x(0) + V_i\right) u_i(k) \right\|$$

$$\le \sum_{k=0}^{N-1} \left(\sum_{i=1}^{p} \left\| A^{N-1-k}\left(B_i A^k x(0) + V_i\right) \right\|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{p} \left\| u_i(k) \right\|^2 \right)^{\frac{1}{2}}$$

$$\le \sum_{k=0}^{N-1} \|T(k)\| \|u(k)\|$$

$$\le \left(\sum_{k=0}^{N-1} \|T(k)\|^2 \right)^{\frac{1}{2}} \left(\sum_{k=0}^{N-1} \|u(k)\|^2 \right)^{\frac{1}{2}}$$

$$\le \alpha \|u\|$$

So H is linear

- For the adjoint operator we have

$$\langle Hu, x \rangle = \langle Pu, x \rangle = \left\langle u, P^T x \right\rangle = \langle u, x \rangle$$

Hence the expression of H^*.

Proposition 3
If H is surjective then (1) is locally controllable.

Proof.
Let $x_0 ; x_d \in \mathbb{R}^n$. Thus $x_d - A^N x_0 \in \mathbb{R}^n$
As H is surjective, there exists a control $u \in L^2\left(0, N-1, \left(\mathbb{R}^p\right)^N\right)$ such as $Hu = x_d - A^N x_0$, so $x_d = A^N x_0 + Hu$, then $x_u^0(N) = x_d$
Hence the result according to the definition 1.

Proposition 4
If rank [P] = n; then the system (1) is locally controllable.

Proof.
Let H be the operator defined by (4) and H^* is the adjoint operator.
We know that $\operatorname{Im} H = \mathbb{R}^n \Leftrightarrow \ker H^* = \{0\}$
If $rank\left[P\right] = rank\left[P_{N-1}\ P_{N-2}\ \cdots P_0\right] = n$

Then $\ker\begin{bmatrix} P_{N-1}^T \\ P_{N-2}^T \\ \vdots \\ P_0^T \end{bmatrix} = \{0\}$

Hence $\ker H^* = \{0\}$
So if *rank [P] = n* then H is surjective and according to the proposition (3), the system (1) is locally controllable.

4. Optimal control

In this section we focus on the characterization of optimal control for the case of system (1).
Let introduce the matrix W defined by

$$W = \sum_{k=0}^{N-1} P_k P_k^T \tag{5}$$

We have the following result

Theorem 5
The system (1) is locally controllable if the matrix P has full rank. Furthermore, the control $u^*(\cdot)$ *which can transfer the system from the initial state* x_0 *to the final state* x_d *with a minimum energy, is given by*

$$\begin{cases} u^*(k) = -P_k^T W^{-1}\left(A^N x(0) - x_d\right) \\ k \in \{0,1,\ldots,N-1\} \end{cases} \tag{6}$$

Before we prove this theorem, we first prove the following lemma:

Lemma 6
The matrix P has full rank if and only if the matrix W is positive definite.

Proof.
$\Rightarrow|$ We have

$$\langle Wx, x\rangle = \left\langle \sum_{k=0}^{N-1} P_k P_k^T x, x \right\rangle = \sum_{k=0}^{N-1} \left\langle P_k^T x, P_k^T x\right\rangle = \left\|P_k^T x\right\|^2 \geq 0$$

If $\langle Wx, x\rangle = 0$ then $P_k^T x = 0$, $\forall k \in \{0,1,\ldots,N-1\}$

Then $x \in \ker\begin{bmatrix} P_{N-1}^T \\ P_{N-2}^T \\ \vdots \\ P_0^T \end{bmatrix} = \{0\}$ because *rank [P] = n*

Hence $x=0$ and therefore W is positive definite.

$\Leftarrow|$ Let $x \in \ker\begin{bmatrix} P_{N-1}^T \\ P_{N-2}^T \\ \vdots \\ P_0^T \end{bmatrix}$

so $P_{N-1}^T x = P_{N-2}^T x = \cdots = P_0^T x = 0$
then $P_k P_k^T x = 0$, $\forall k \in \{0,1,\ldots,N-1\}$ and $Wx = 0$
hence $x=0$ (because W is positive definite)

thus $\ker\begin{bmatrix} P_{N-1}^T \\ P_{N-2}^T \\ \vdots \\ P_0^T \end{bmatrix} = \{0\}$

and finally we get *rank [P] = n*

Proof (of theorem 5).
• First, suppose the matrix P has full rank then, according to the proposition 4, the system (1) is locally controllable. Furthermore, using the previous lemma, the matrix W is positive definite which implies it inversibility. Consequently u^* defined by (6) is well defined.

• Then by replacing u^* in (3) by the expression (6) one can easily check that $\bar{x}(N) = x_d$.

• Finally we show that $\left\|u^*\right\| = \inf U$, with U={ $\|v\|$ / v is a control that allows the transfer of the system from x_0 to x_d }.

Let $u \in U$, then

$$A^N x(0) + \sum_{k=0}^{N-1}\sum_{i=1}^{p} A^{N-1-k}\left(B_i A^k x(0) + V_i\right) u_i(k) = x_d$$

then

$$\sum_{k=0}^{N-1}\sum_{i=1}^{p} A^{N-1-k}\left(B_i A^k x(0) + V_i\right)\left(u_i(k) - u_i^*(k)\right) = 0$$

$$\Rightarrow \left\langle \sum_{k=0}^{N-1}\sum_{i=1}^{p} A^{N-1-k}\left(B_i A^k x(0) + V_i\right)\left(u_i(k) - u_i^*(k)\right); -W^{-1}\left(A^N x(0) - x_d\right)\right\rangle = 0$$

$$\Rightarrow \sum_{k=0}^{N-1}\left\langle P_k\left(u(k) - u^*(k)\right); -W^{-1}\left(A^N x(0) - x_d\right)\right\rangle = 0$$

$$\Rightarrow \sum_{k=0}^{N-1}\left\langle \left(u(k) - u^*(k)\right); -P_k^T W^{-1}\left(A^N x(0) - x_d\right)\right\rangle = 0$$

$$\Rightarrow \sum_{k=0}^{N-1}\left\langle \left(u(k) - u^*(k)\right); u^*(k)\right\rangle = 0$$

$$\Rightarrow \left\langle u - u^*; u^*\right\rangle = 0$$

$$\Rightarrow \left\langle u; u^*\right\rangle - \left\|u^*\right\|^2 = 0$$

$$\Rightarrow \left\|u^*\right\|^2 = \left\langle u; u^*\right\rangle$$

$$\Rightarrow \left\|u^*\right\|^2 \le \left\|u\right\|\left\|u^*\right\|$$

$$\Rightarrow \left\|u^*\right\| \le \left\|u\right\|$$

Hence the result.

5. Example

Consider the dynamical system (Mohler, 1973)

$$\dot{x} = Ax + u_1 B_1 + Bu \qquad (7)$$

Where

$$A = \begin{bmatrix} \frac{-R_a}{L_a} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{D}{J} \end{bmatrix}; \quad B_1 = \begin{bmatrix} 0 & 0 & \frac{-K_a}{L_a} \\ 0 & 0 & 0 \\ \frac{K_y}{J} & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \frac{1}{L_a} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} i_a \\ \theta \\ \omega \end{bmatrix} \text{ and } u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} i_e \\ \upsilon_a \end{bmatrix}$$

J is the moment of inertia, D is the viscous damping ratio, R_a is the armature resistance, L_a is the applied armature inductance, K_y , K_a are motor characteristics, K_a is the motor const, i_a is the armature current, i_e is the field current, υ_a is the armature voltage, ω is the angular velocity, and θ is the angular position.

Equation (7) can be discretized by use of a first-order Euler expansion to give

$$x(k+1) = x(k) + TAx(k) + u_1(k)TB_1 x(k) + TBu(k) \qquad (8)$$

where T is the sampling interval. Equation (8) can be rewritten as

$$x(k+1) = A^* x(k) + u_1(k) B_1^* x(k) + B^* u(k) \qquad (9)$$

With $A^* = I + TA$, $B_1^* = TB_1$ and $B^* = TB$

The parameter values chosen for the model are taken from [9] and are T = 0.1, K_a =0.156, K_y =37.7, L_a =0.05, $J = 2.4 \times 10^{-4}$, D = 0.0032 and.
Then the system (9) becomes

$$x(k+1) = \begin{bmatrix} 0.880 & 0 & 0 \\ 0 & 1 & 0.1 \\ 0 & 0 & -0.334 \end{bmatrix} x(k) + u_1(k)\begin{bmatrix} 0 & 0 & -75.4 \\ 0 & 0 & 0 \\ 15708.334 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(k) \qquad (10)$$

Consider $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ the initial state and $x_d = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$ the desired state.

We present numerical results obtained using Matlab.
For N = 20; we have *rank[P]* = 3, so the system (10) is locally controllable and the optimal control is given by the following table.

Table 1: optimal control

k	$u_1(k)$	$u_2(k)$
0	-0.864	0.028
1	0.554	0.032
2	-0.011	0.036
3	0.179	0.041
4	0.086	0.047
5	0.103	0.053
6	0.080	0.060
7	0.074	0.068
8	0.064	0.078
9	0.057	0.088
10	0.050	0.100

11	0.044	0.114
12	0.039	0.130
13	0.034	0.147
14	0.030	0.168
15	0.026	0.190
16	0.023	0.216
17	0.020	0.246
18	0.019	0.280
19	0.012	0.318

6. Conclusion

In this paper, we have studied the local controllability of a bilinear discrete-time system. The method that we present in this paper is based on a linearization of the system and then the definition of a suitable operator which can lead to control transferring the system to a desired given state with a minimum energy.

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