Research on Two Algorithms of Solving Large-scale Tridiagonal Linear Equations

Yu Bencheng[1],Chen Yan[2]

[1]**Information and management institute of technology, Xuzhou college of industrial technology ,Xuzhou, China**

[2]**Information and management institute of technology, Xuzhou college of industrial technology ,Xuzhou, China**

Abstract

Based on the analysis of the two kinds of algorithms in solving large-scale tridiagonal linear equations, which are linear interpolation method and the method of double parameters, it is shown that the principle of the linear interpolation method and double parameter method is consistent and it points out that in this principle, the solutions to certain types of tridiagonal equations in the two methods are not stable. But in the case of not so sick, their relative errors of solution are very small, and the situation is very stable.

Key Words: *Tridiagonal Linear Equations, Complexity of algorithm, Stability, Algorithm.*

1. Introduction

There are many applications of tridiagonal equations whose general forms are shown as follows:

$$A_x = f \tag{1}$$

Among which:

$$A=\left[a_{i-1},b_i,c_i\right]_{i-1}^{n}=\begin{pmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_{n-1} & b_n \end{pmatrix} \tag{2}$$

$f=(f_1, f_2,\cdots, f_n)^T$, $a_i c_i \neq 0\left(i=1,2,\cdots,n-1\right)$ .When A is a singular matrix[1], it has a unique solution to equations $A_x = f$.

For such equations, of course, we can use the Gaussian elimination method [2]. But when matrix dimension is very big, the calculation amount by the method of Gaussian elimination will be too large [3]. For its special structure, we can construct some special solutions so as to reduce the amount of calculation. There are a variety of special solutions to tridiagonal equations [4]. This paper mainly introduced two kinds of algorithms and compared in detail the calculation of the two algorithms. Based on the principle of the two methods, it pointed out the similarity between these two algorithms and the instability in the solutions to certain kind of equations.

2. Linear Interpolation Method

Linear interpolation process is described as follows:

The former n-1 of equations of $A_x = f$ is $A_{n-1}{}^{x} = f^{'}$.

f' is made up of the former $n-1$ element of f'.

The latter $n-1$ column vector of A_{n-1} is $\tilde{A}$, among which $\tilde{A}$ is a lower triangular matrix of $c_i\,(i=1,\ldots,n-1)$.

For $c_i \neq 0(i=1,2,\cdots,n-1)$, $\tilde{A}$ is a non-singular matrix.

As long as x_1 is given, we can get $\tilde{x}=(x_2,\cdots,x_n)^T$ by

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solving $\tilde{A}\tilde{x}$, among which $\tilde{f}=(f_1-b_1x_1, f_2-a_1x_1, f_3,\cdots, f_{n-1})^T$.

Thus we can get $x=(x_1,x_{2,}\cdots,x_n)^T$, which is a set of solutions to $A_{n-1}x=\tilde{f}$. Then two sets of different solutions are get by taking two different x_1, and the solution of equation $A_x=f$ is obtained through the linear combination of the two sets of solutions [5].

Assuming $x_1^{(0)}=0$, by the solution of $\tilde{A}\tilde{x}^{(0)}=\tilde{f}$, we can get $x^{(0)}x=(x_1^{(0)},x_2^{(0)},\cdots,x_n^{(0)})^T$ which satisfies $A_{n-1}x^{(0)}=f'$. And similarly assuming $x_1^{(1)}=1$, $x^{(1)}=(x_1^{(1)},x_2^{(1)},\cdots,x_n^{(1)})^T$ is get which satisfies $A_{n-1}x^{(1)}=f'$ [6].

For $\forall h\in C$, C is complex number field, $A_{n-1}\left[x^{(0)}+h(x^{(1)}-x^{(0)})\right]=f'$. As long as h is selected to make $x=x^{(0)}+h(x^{(1)}-x^{(0)})$ satisfy the n^{th} equation of $Ax=f$, the solution of $Ax=f$ is obtained and it is easily to get

$$h=\frac{f_n-a_{n-1}x_{n-1}^{(0)}-b_nx_n^{(0)}}{a_{n-1}(x_{n-1}^{(1)}-x_{n-1}^{(0)})+b_n(x_n^{(1)}-x_n^{(0)})}.$$

3. Double Parameter Method

Double parameter method is slightly different for that x_1 is used as the parameter to determine the relationship between the other variables and x_1, then take a substitution equations $A_x=f$ to determine x_1, thus a solution to $A_x=f$ is obtained [7]. The process is described as follows:

$\tilde{x}$ (Shown as x_1)is calculated by $\tilde{A}\tilde{x}=\tilde{f}$:

$$\begin{cases} x_2=(f_1-b_1x_1)/c_1 \\ x_3=(f_2-a_1x_1-b_2x_2)/c_2 \\ \vdots \\ x_n=(f_{n-1}a_{n-2}x_{n-2}-b_{n-1}x_{n-1})/c_{n-1} \end{cases} \tag{3}$$

Take substitutions in a sequence series to change them into a form of containing only constant term and x_1.

Assuming $s_1=0, t_1=1$, then $x_1=s_1+t_1x_1$, so we can get

$$x_i=s_i+t_ix_1(i=2,3,...,n) \tag{4}$$

Among which $s_i,t_i(i=2,...,n)$ is undetermined parameters. Comparing $x_i=s_i+t_ix_1(i=2,3,...,n)$ and

$$\begin{cases} x_2=(f_1-b_1x_1)/c_1 \\ x_3=(f_2-a_1x_1-b_2x_2)/c_2 \\ \vdots \\ x_n=(f_{n-1}a_{n-2}x_{n-2}-b_{n-1}x_{n-1})/c_{n-1} \end{cases} \tag{5}$$

We can get recurrence relations of parameter group :

$$\begin{aligned} & s_i,t_i,i=1,....,n. \\ & s_1=0, s_2=f_1/c_1 \\ & s_i=(f_{i-1}-b_{i-1}s_{i-1}-a_{i-2}s_{i-2})/c_{i-1}(i=3,4,...,n) \\ & t_1=1, t_2=-b_1/c_1 \\ & t_i=-(b_{i-1}t_{i-1}+a_{i-2}t_{i-2})/c_{i-1}(i=3,4,...,n) \end{aligned}$$

According to the n equation of the equation group $A_x=f$, the way of expression of x_1 can be obtained:

$$x_1=\frac{f_n-a_{n-1}s_{n-1}-b_ns_n}{a_{n-1}t_{n-1}+b_nt_n} \tag{6}$$

Inserting $x_i=s_i+t_ix_1(i=2,3,...,n)$, we can get a solution to $A_x=f$.

4. The relationship between Linear interpolation method and the method of double parameters

Through the comparison of the two methods, we can find the internal relationship among the parameters [8]. From the expression $x_i=s_i+t_ix_1(i=2,3,...,n)$, assuming x_1

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is 0 and 1 respectively, we can get $x^{(0)} = (s_1, s_2, ..., s_n)^T$

$x^{(1)} = (s_1 + t_1, s_2 + t_2, ..., s_n + t_n)^T$,

in which $x^{(0)}, x^{(1)}$ are shown in linear interpolation method and is shown in double parameter method. The relationship between the two methods is described as below [9]:

Linear Interpolation Double Parameter Method

$$x^{(0)} \Longleftrightarrow s$$

$$x^{(1)} - x^{(0)} \Longleftrightarrow t$$

$$h \Longleftrightarrow x_1$$

Take double parameters method for example to go into its computational complexity. 3 n - 5 times multiplying and dividing is needed to calculate s and t, while n-1+5=n+4 times multiplying and dividing is needed to calculate x, so in total 2（3n-5）+n+4=7n-6 times multiplying and dividing is needed. So its computational complexity is: O (7 n).

The above two methods both require the solution to $\tilde{A}\tilde{x} = \tilde{f}$, so the stability of the two methods is closely related to the condition number of $\tilde{A}$. If $\tilde{A}$ is a morbid matrix, a lot of errors may appear by using these two methods to get the numerical solution. These two kinds of methods for tridiagonal equations can be used, but there may be numerical instability for some problems.

5. Numerical Example

Example： A_n is shown as below:：

$$A_n = \begin{pmatrix} 3 & 1 & & & \\ 1 & 3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 3 & 1 \\ & & & 1 & 3 \end{pmatrix}_n \quad (7)$$

Among them, n represents a matrix dimension [10].

The selection of the right end item f is like that, given a n column vector y at random, then left multiply A to get f.

The table below lists the relative error of various methods.

The formula of Relative error is $\in = \|y - x\|_2 / \|y\|_2$,

among which x is the solution by various methods and y is the exact solution of equations set。

Relative error of various methods is shown in table 1.

Table 1. Relative Error

n	chasing	linear interpolation	biparameter	QR decomposition
16	1.9442 E-16	6.3124 E-11	7.6431 E-11	4.3368 E-16
32	2.6459 E-16	2.0342 E-04	3.7341 E-05	4.4902 E-16
64	9.4034 E-17	1.6614 E+10	2.1380 E+09	3.8538 E-16
128	2.0887 E-16	2.3835 E+36	1.5808 E+35	3.0555 E-16
256	2.2368 E-16	*	*	3.3981 E-16
512	2.2412 E-16	*	*	3.4561 E-16
1024	2.2935 E-16	*	*	3.1418 E-16

As seen from the table, although the conditions of A is little (when n=1024, $\|A\|_2 < 5$), the result got from the linear interpolation method and biparameter method may not be so good.

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The equations of the lower triangular $\tilde{A}\tilde{x}=\tilde{f}$ should also be required to be solved. Its coefficient matrix $\tilde{A}$ is *n* - 1 order matrix after getting rid of the first column and the last line of matrix *A*. So the stability of the two methods has something with the condition numbers of *A*.

In this example, when *n*=32,the condition numbers of $\tilde{A}$ are 6.1989E+13，and when *n*=64，they are up to as high as 1.2721E+17. So it is obvious that it is a sick equation, which leads to the instability of the linear interpolation method. So when the two methods of the linear interpolation method and biparameter method are used, it is required to check if the condition numbers of $\tilde{A}$ are small enough.

6. Conclusion

Through the contrast between the linear interpolation method, biparameter method and chasing method, QR decomposition method, it is apparent that as long as the solution of the problem is not sick, the relative errors of the solutions are very small and stable. In other cases, the calculation speed and algorithm complexity of double parameter method and the linear interpolation method are small, so in actual application process the two methods are very valuable.

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