Multi-period Optimal Portfolio Decision with Transaction Costs and HARA Utility Function

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Abstract

Portfolio selection problem is one of the core research fields in modern financial management. While considering the transaction costs in the long term investment makes the portfolio selection problems more complex than there are no transaction costs. In this paper, the general multi-period investment problems with HARA utility function and proportional transaction costs are investigated. By using the dynamic programming method, the indirect utility function is defined for solving the portfolio selection problem. The optimal strategies and the boundary of the no-transaction region are obtained in the explicit form. And the procedure for solving the original portfolio selection problem is given. Numerical example shows the feasibility and effectiveness of the method provided in this paper.

Keywords: *Optimal portfolio, Dynamic programming, Transaction costs, HARA utility function.*

1. Introduction

The portfolio selection problem is one of the most important problems faced to the investors, who need to allocate his or her wealth among different assets or assets classes properly. Determining the optimal portfolio is a rather complex problem which depends on the objective of the investor. In the single period setting, the problem is well understood and can be easily solved by using the mean-variance model [1] or other static models (see [2]). In the multi-period setting, the problem is more complex than the single period one. The multi-period portfolio problem was proposed by [3] and [4]. Explicit solutions for these problems are only available under some assumptions: investment opportunities are constant; there are no transaction costs; the short sale is allowed and the market is complete.

It is well known that an investor who ignores the transaction costs would end up bankrupt. Several authors have made important contributions to the effect of the transaction costs in the multi-period setting (see, for example, [5-18]). Kamin [5] introduced the transaction cost into the dynamic portfolio selection model, and found that the investor's behavior is systematically different from the one without transaction costs. Constantinides [6] extended Kamin's model to the HARA utility function. Magill and Constantinides [7] developed a method to determine the impact of trading costs on capital market equilibrium. Constantinides [8] showed that in the case of proportional transaction costs and power utility, the no-transaction region is of great importance for all practical applications, and believed that these boundaries cannot be obtained analytically. Then, he developed approximate solutions for the case of the investor with a power utility (see [9]). In [10], they studied the optimal consumption and investment decision with the transaction costs for an investor and gave an algorithm for solving the free boundary problem.

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These solutions usually deal with the case in an infinite time horizon. But it is more realistic to analyze a finite terminal time. In this case, Gennotte and Jung [11] developed a numerical approximate value of the boundaries. Akian et al. [12] considered the n risky assets situation, and gave the viscosity solution. Boyle and Lin [13] extended Gennotte and Jung's work, and illustrated the solution procedure in which the returns on the risky asset follow a multiplicative binomial process. Framstad et al. [14] showed that the solution in a jump diffusion market has the same form as in the pure diffusion case. Jang [15] investigated an optimal portfolio selection problem with transaction costs when an illiquid asset pays cash dividends and there are constraints on the illiquid asset holding, and provided the closed form solutions for the problem.

Motivated by the above results, we extend research by Boyle and Lin to include the case where the investor has the HARA utility functions. We provide an explicit closed form solution to the finite horizon problem when there are proportional transaction costs and the investor has the HARA utility function. A procedure to derive the boundaries of the no-transaction region is also given.

2. THE MODEL

2.1 HARA utility function

The definition of the general class of HARA utility function is introduced in this subsection. This kind of utility function is very general indeed since it contains the most used utility functions.

Definition 1. *A utility function U is said to have harmonic absolute risk aversion (HARA) if the inverse of its absolute risk aversion is linear in wealth.*

Remark 1. According to Definition 1, the HARA utility function can be written as:

$$U(x) = a \cdot \left(b + \frac{x}{c}\right)^{1-c} \tag{1}$$

with the domain $b + \frac{x}{c} > 0$. The constant parameters a, b, and c satisfy the condition: $a(1-c) > 0$.

Usually four subclasses are distinguished. When $c = -1$, the utility function is the quadratic utility function. As $c \to \infty$, the utility function takes the form

$$U(x) = ab \cdot \exp\left(-\frac{x}{b}\right)^{1-c},$$

which is often called the CARA (Constant Absolute Risk Aversion) or exponential utility function. If $b = 0$, $c \neq 1$ the utility function is the CRRA (Constant Relative Risk Aversion) or power utility function, which formed

$$U(x) = a \cdot \frac{x^{(1-c)}}{1-c}.$$

Because of $\lim_{c \to 0}(x^c - 1) \cdot \frac{1}{c} = \ln x$, $U(x) = \ln x$ can be considered as another special case of HARA utility function.

2.2 Dynamic programming with utility functions

Consider a financial market where an investor can make decision for his sequential investment at T trading times, indexed as $t = 1, 2, \cdots, T$, over a finite planning period. There are two securities: one riskless asset and one risky asset at each time. Denote P_0^t the price of the riskless asset and P_t the prices of the risky assets at time t. For $t = 1, 2, \cdots, T-1$, $r^0 = \frac{P_{t+1}^0}{P_t^0}$ is the total return on riskless asset and $r_t = \frac{P_{t+1}}{P_t}$ is the total return on the risky assets respectively. Thus, r^0 is a constant and r_t is a random variable.

Assume that an investor holds a portfolio with $x_1^0 \geq 0$ dollars of the riskless asset and $x_1 \geq 0$ dollars of the risky asset at the initial time $t = 1$. At each trading time $t = 1, 2, \cdots, T-1$, the investor may make his investment decision to maximize his expected utility of terminal wealth. Let x_t^0 be the dollar amounts of the riskless asset and x_t be dollar amounts of the risky asset in the portfolio at time t before trading. It is assumed that there is a transaction cost proportional to the amount of

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each risky asset traded. Let θ be the unit transaction cost for buying or selling the risky asset. We use u_t to denote investment decision at time t. u_t is the amount of the risky asset traded, $u_t \geq 0$ for buying and $u_t \leq 0$ for selling. Thus, the total transaction costs could be $\theta \cdot |u_t|$. Then, the following relationships can be built:

$$y_t^0 = x_t^0 - u_t - \theta \cdot |u_t|, \tag{2}$$

$$y_t = x_t + u_t, \tag{3}$$

here y_t^0 is the dollar amounts of the riskless asset, y_t is the dollar amounts of the risky asset at time t after trading. We also assume that y_t^0 and y_t are non-negative.

Thus, at time $t+1$ the portfolio amounts before trading can be written as:

$$x_{t+1}^0 = y_t^0 r^0 = (x_t^0 - u_t - \theta \cdot |u_t|) r^0, \tag{4}$$

$$x_{t+1} = y_t r_t = (x_t + u_t) r_t. \tag{5}$$

Equation (4) and (5) describe feasible investment decisions. The objection is to find an optimal sequential investment strategy that maximizes the expected utility of terminal wealth, namely:

$$\max_{u_t, t=1,2,\cdots,T-1} \quad E[U(x_T^0, x_T)], \tag{6}$$

for the given initial portfolio (x_1^0, x_1). Here $U(\cdot)$ represents the HARA utility function.

From above preparation, the model for the investment problem can be presented as:

$$\begin{array}{ll} \max\limits_{u_t, t=1,2,\cdots,T-1} & E[U(x_T^0, x_T)] \\ subject \;\; to & x_{t+1}^0 = (x_t^0 - u_t - \theta \cdot |u_t|) r^0, \\ & x_{t+1} = (x_t + u_t) r_t, \\ & t = 1,2,\cdots,T-1. \end{array} \tag{7}$$

Problem (7) can be solved by a dynamic programming technique. In [13], it was assumed that the terminal utility function U must be a concave, homogeneous differentiable function with some degree, say α. As we assumed above, the terminal utility function U is the HARA utility function which taken the form of (1) is a concave and differentiable function to the terminal total wealth. But it is not homogeneous. Thus we need to do the following transformations. Let

$$\begin{array}{l} \bar{x}_T^0 = x_T^0 + bc, \\ \bar{x}_t^0 = x_t^0 + bc \cdot \left(\dfrac{1}{r^0}\right)^{T-t}, \qquad t = 1,2,\cdots,T-1, \end{array} \tag{8}$$

Thus,

$$U(x_T^0, x_T) = \tilde{U}(\bar{x}_T^0, x_T) = a\left(\frac{\bar{x}_T^0 + x_T}{c}\right)^{1-c}. \tag{9}$$

Then, problem (7) is equivalent to problem (10):

$$\begin{array}{ll} \max\limits_{u_t, t=1,2,\cdots,T-1} & E[\tilde{U}(\bar{x}_T^0, x_T)] \\ subject \;\; to & \bar{x}_{t+1}^0 = (\bar{x}_t^0 - u_t - \theta \cdot |u_t|) r^0, \\ & x_{t+1} = (x_t + u_t) r_t, \\ & t = 1,2,\cdots,T-1. \end{array} \tag{10}$$

To apply dynamic programming, we define the indirect utility function V_t, $t = 1,2,\cdots,T$, as follows:

$$V_t(\bar{x}_T^0, x_T) = \begin{cases} \tilde{U}(\bar{x}_T^0, x_T), & t = T, \\ \max\limits_{u_t} E_t V_{t+1}(\bar{x}_{t+1}^0, x_{t+1}), & t = 1,2,\cdots,T-1. \end{cases} \tag{11}$$

Here, E_t denotes the expectation over r_t conditional on x_t^0 and x_t. According to the Bellman principle of optimality, the variable u_t, which maximizes $E_t V_{t+1}(\bar{x}_{t+1}^0, x_{t+1})$, $t = 1,2,\cdots,T-1$, forms the optimal trading strategy of the problem (7).

3. OPTIMAL STRATEGY

In this section, we develop the procedures for solving problem (7) and (10). The derivation is based on the main theorem of [13].

Let

$$\begin{aligned} g_t(u_t, \bar{x}_t^0, x_t) &= E_t V_{t+1}(\bar{x}_{t+1}^0, x_{t+1}) \\ &= E_t V_{t+1}((\bar{x}_t^0 - u_t - \theta \cdot |u_t|) r^0, (x_t + u_t) r). \end{aligned} \tag{12}$$

Then, we give the definition of the no-transaction region. For any portfolio in this region, the expected value will not be increased by buying or selling the risky asset.

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Definition 2. When the set of portfolio Φ_t satisfies

$$\Phi_t = \left\{ \left(\overline{x}_t^0, x_t\right) \mid g_t\left(u_t, \overline{x}_t^0, x_t\right) \le g_t\left(0, \overline{x}_t^0, x_t\right), \text{ for all } u_t \right\} \quad (13)$$

Φ_t is called the no-transaction region at time t.

Let $\partial^+ g_t / \partial u_t$, $\partial^- g_t / \partial u_t$ denote the right and left derivatives of g_t respectively. According to the main theorem of [13], if $0 < a_t \le b_t < \infty$, the no-transaction region can be written as:

$$\Phi_t = \left\{ \left(\overline{x}_t^0, x_t\right) \mid a_t \le x_t / \overline{x}_t^0 \le b_t \right\},$$

where

$$a_t = \min\left\{ x_t \mid \frac{\partial^+ g_t(0,1,x_t)}{\partial u_t} = 0, x_t \ge 0 \right\},$$

$$b_t = \max\left\{ x_t \mid \frac{\partial^- g_t(0,1,x_t)}{\partial u_t} = 0, x_t \ge 0 \right\}.$$

The optimal transaction strategies for problem (10) are given in the bellowing theorem.

Theorem 1. *If* $a_t \le x_t / \overline{x}_t^0 \le b_t$*, then there will be no buying or selling the risky asset.*

If $a_t > x_t / \overline{x}_t^0$ *, then*

$$\max_{u_t} g_t(u_t, \overline{x}_t^0, x_t) = g_t(u_t^+, \overline{x}_t^0, x_t) = g_t(0, y_t^{0+}, y_t^+), \quad (14)$$

where

$$\begin{aligned} u_t^+ &= \frac{\overline{x}_t^0 a_t - x_t}{1 + (1+\theta) a_t}, \\ y_t^{0+} &= \overline{x}_t^0 - (1+\theta) u_t^+, \\ y_t^+ &= x_t + u_t^+, \end{aligned} \quad (15)$$

with $(y_t^{0+}, y_t^+) \in \Phi_t$ *and* $y_t^+ / y_t^{0+} = a_t$.

If $b_t < x_t / \overline{x}_t^0$ *, then*

$$\max_{u_t} g_t(u_t, \overline{x}_t^0, x_t) = g_t(u_t^-, \overline{x}_t^0, x_t) = g_t(0, y_t^{0-}, y_t^-), \quad (16)$$

where

$$\begin{aligned} u_t^- &= \frac{\overline{x}_t^0 b_t - x_t}{1 + (1-\theta) b_t}, \\ y_t^{0-} &= \overline{x}_t^0 - (1-\theta) u_t^-, \\ y_t^- &= x_t + u_t^-, \end{aligned} \quad (17)$$

with $(y_t^{0-}, y_t^-) \in \Phi_t$ *and* $y_t^- / y_t^{0-} = b_t$.

Proof. Case 1, $a_t < \infty$. since $u_t^+ > 0$,

$$\begin{aligned} &\frac{\partial g_t(u_t^+, \overline{x}_t^0, x_t)}{\partial u_t} \\ &= -r^0 (1+\theta) E_t \frac{\partial V_{t+1}\left((\overline{x}_t^0 - (1+\theta)\cdot u_t^+) r^0, (x_t + u_t^+) r_t \right)}{\partial \overline{x}_t^0} \\ &\quad + E_t r_t \frac{\partial V_{t+1}\left((\overline{x}_t^0 - (1+\theta)\cdot u_t^+) r^0, (x_t + u_t^+) r_t \right)}{\partial x_{t+1}} \\ &= (\overline{x}_t^0 - (1+\theta)\cdot u_t^+)^{-c} \cdot \left\{ -r^0(1+\theta) E_t \frac{\partial V_{t+1}\left(r^0, a_t r_t\right)}{\partial \overline{x}_t^0} \right. \\ &\quad \left. + E_t r_t \frac{\partial V_{t+1}\left(r^0, a_t r_t\right)}{\partial x_{t+1}} \right\} \\ &= (\overline{x}_t^0 - (1+\theta)\cdot u_t^+)^{-c} \cdot \frac{\partial g_t^+\left(0,1,a_t\right)}{\partial u_t}. \end{aligned}$$

As we mention above that $\frac{\partial^+ g_t(0,1,a_t)}{\partial u_t} = 0$, thus

$$\frac{\partial^+ g_t(u_t^+, \overline{x}_t^0, x_t)}{\partial u_t} = 0. \quad (18)$$

It is shown that u_t^+ is a maximum point.

Case 2, $a_t = \infty$. As g_t is non-decreasing, u_t^+ is the right end point of its domain, thus u_t^+ is the maximum point.

Similarly, u_t^- is the maximum point when $b_t < x_t / \overline{x}_t^0$. □

Thus, Theorem 2 gives the optimal strategies of the original problem (7) below.

Theorem 2. *If* $a_t \le \frac{x_t}{x_t^0 + bc(1/r^0)^{T-t}} \le b_t$*, then* $\overline{u}_t = 0$*, i.e. there will be no buying or selling the risky asset.*

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If $a_t > \dfrac{x_t}{x_t^0 + bc(1/r^0)^{T-t}}$, *then*

$$\begin{aligned}\tilde{u}_t^+ &= \frac{x_t^0 a_t - x_t + bc(1/r^0)^{T-t}\cdot a_t}{1+(1+\theta)a_t},\\ \tilde{y}_t^{0+} &= x_t^0 + bc(1/r^0)^{T-t} - (1+\theta)\tilde{u}_t^+,\\ y_t^+ &= x_t + \tilde{u}_t^+,\end{aligned} \tag{19}$$

with $(\tilde{y}_t^{0+}, \tilde{y}_t^+) \in \Phi_t$ *and* $\tilde{y}_t^+ / \tilde{y}_t^{0+} = a_t$.

If $b_t < \dfrac{x_t}{x_t^0 + bc(1/r^0)^{T-t}}$, *then*

$$\begin{aligned}\tilde{u}_t^- &= \frac{x_t^0 b_t - x_t + bc(1/r^0)^{T-t}\cdot b_t}{1+(1-\theta)b_t},\\ \tilde{y}_t^{0-} &= x_t^0 + bc(1/r^0)^{T-t} - (1-\theta)\tilde{u}_t^-,\\ y_t^- &= x_t + \tilde{u}_t^-,\end{aligned} \tag{20}$$

with $(\tilde{y}_t^{0-}, \tilde{y}_t^-) \in \Phi_t$ *and* $\tilde{y}_t^- / \tilde{y}_t^{0-} = b_t$.

Proof. We know that $\overline{x}_t^0 = x_t^0 + bc(1/r^0)^{T-t}$, $t = 1, 2, \cdots, T$. Then, substituting $\overline{x}_t^0 = x_t^0 + bc(1/r^0)^{T-t}$ into (15) and (17), we will get the optimal strategies of the original problem (7). □

Now, how to calculate a_t, b_t and the indirect utility function are presented.

Definition 3. *V is a piece-wise linear utility function with respect to the function U, if there is a sequence of increasing numbers q_j, $j = 1, 2, \cdots, s$, and non-negative constants α_{ij} and β_{ij} with respect to the underlying probability space $\{\omega_i; i = 1, 2, \cdots, I\}$ such that*

$$V(x^0, x) = \sum_{i=1}^{I} U(\alpha_{ij}x^0, \beta_{ij}x)\Pr(\omega_i), \tag{21}$$

$$\text{for } q_j \le x/x_0 \le q_{j+1}. \tag{22}$$

Assuming that V_t is a piecewise linear utility function with respect to U. Let $\{r_t^k; k = 1, 2, \cdots, K\}$ be all possible outcomes for r_t, and $\{\omega_i\}$ be all possible outcomes from $(r_{t+1}, r_{t+2}, \cdots, r_T)$, where $\{\omega_i\}$ represents the set of all future paths of the underlying tree structure starting at a node at time $t+1$. Then starting at time t all the paths of the underlying tree structure can be written as $\{(r_t^k, \omega_i)\}$.

Now, calculate V_t recursively starting at $t = T$. At $t = T$,

$$V_T(\overline{x}_T^0, x_T) = U(\overline{x}_T^0, x_T) = a\left(\frac{\overline{x}_T^0 + x_T}{c}\right)^{1-c}. \tag{23}$$

Suppose that

$$\begin{aligned}V_{t+1}(\overline{x}_{t+1}^0, x_{t+1}) &= \sum_{i=1}^{I} U(\alpha_{ij}\overline{x}_{t+1}^0, \beta_{ij}x_{t+1})\Pr\{\omega_i\}\\ &= \frac{a}{c^{1-c}}\sum_{i=1}^{I}(\alpha_{ij}\overline{x}_{t+1}^0 + \beta_{ij}x_{t+1})^{1-c}\Pr\{\omega_i\},\end{aligned} \tag{24}$$

$$q_j \le \frac{x_{t+1}}{\overline{x}_{t+1}^0} \le q_{j+1}, j = 0, 1, \cdots, s; q_0 = 0, q_{s+1} = \infty.$$

Then,

$$\begin{aligned}g_t(0, \overline{x}_t^0, x_t) &= E_t V_t(\overline{x}_t^0 r^0, x_t r_t)\\ &= \sum_{k=1}^{K}\sum_{i=1}^{I} U(\alpha_{ij}r^0\overline{x}_t^0, \beta_{ij}r_t^k x_t)\Pr\{(r_t^k, \omega_i)\}\\ &= \frac{a}{c^{(1-c)}}\sum_{k=1}^{K}\sum_{i=1}^{I}(\alpha_{ij}r^0\overline{x}_t^0 + \beta_{ij}r_t^k x_t)^{1-c}\Pr\{(r_t^k, \omega_i)\}.\end{aligned} \tag{25}$$

Let $\tilde{\alpha}_{ij} = \alpha_{ij}r^0$, $\tilde{\beta}_{ij} = \beta_{ij}r_t^k$, thus, when $u_t \ge 0$,

$$\begin{aligned}&g_t(u_t, \overline{x}_t^0, x_t)\\ &= \frac{a}{c^{(1-c)}}\sum_{k=1}^{K}\sum_{i=1}^{I}\left\{\tilde{\alpha}_{ij}[\overline{x}_t^0 - (1+\theta)u_t] + \tilde{\beta}_{ij}[x_t + u_t]\right\}^{1-c}\Pr\{(r_t^k, \omega_i)\};\end{aligned} \tag{26}$$

and when $u_t \le 0$,

$$\begin{aligned}&g_t(u_t, \overline{x}_t^0, x_t)\\ &= \frac{a}{c^{(1-c)}}\sum_{k=1}^{K}\sum_{i=1}^{I}\left\{\tilde{\alpha}_{ij}[\overline{x}_t^0 - (1-\theta)u_t] + \tilde{\beta}_{ij}[x_t + u_t]\right\}^{1-c}\Pr\{(r_t^k, \omega_i)\}.\end{aligned} \tag{27}$$

Therefore, a_t is a solution of one of the equations

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$$\sum_{k=1}^{K}\sum_{i=1}^{I}(\tilde{\alpha}_{ij}+\tilde{\beta}_{ij}a_t)^{-c}[\tilde{\beta}_{ij}-(1+\theta)]\tilde{\alpha}_{ij}\Pr\left\{(r_t^k,\omega_i)\right\}=0, \quad \tilde{q}_j \le a_t < \tilde{q}_{j+1}, j=0,1,2\cdots \tag{28}$$

and b_t is a solution of one of the equations

$$\sum_{k=1}^{K}\sum_{i=1}^{I}(\tilde{\alpha}_{ij}+\tilde{\beta}_{ij}a_t)^{-c}[\tilde{\beta}_{ij}-(1-\theta)]\tilde{\alpha}_{ij}\Pr\left\{(r_t^k,\omega_i)\right\}=0, \quad \tilde{q}_j \le a_t < \tilde{q}_{j+1}, j=0,1,2\cdots \tag{29}$$

The indirect utility function can be calculated as follows: Rearrange all $\left(r^0/r_t^k\right)q_j$, $k=1,2\cdots,K$, $j=0,1,2\cdots$, from smallest to largest and relabeled them as $\tilde{q}_h$ in order of magnitude. Thus, for $l=1,2,\cdots,I$, $\tilde{\alpha}_{lh}=\alpha_{lj}r^0$, $\tilde{\beta}_{lh}=\beta_{lj}r_t^1$,where $\left(r^0/r_t^1\right)q_j \le \tilde{q}_h \le \tilde{q}_{h+1} \le \left(r^0/r_t^1\right)q_{j+1}$; for $l=I+1,I+2,\cdots,2I$, $\tilde{\alpha}_{lh}=\alpha_{l-I,j}r^0$, $\tilde{\beta}_{lh}=\beta_{l-Ij}r_t^2$, where $\left(r^0/r_t^2\right)q_j \le \tilde{q}_h \le \tilde{q}_{h+1} \le \left(r^0/r_t^2\right)q_{j+1}$; … ; for $l=(K-1)I+1,(K-1)I+2,\cdots,KI$, $\tilde{\alpha}_{lh}=\alpha_{l-(K-1)I,j}r^0$, $\tilde{\beta}_{lh}=\beta_{l-(K-1)I,j}r_t^K$, where

$\left(r^0/r_t^K\right)q_j \le \tilde{q}_h \le \tilde{q}_{h+1} \le \left(r^0/r_t^K\right)q_{j+1}$.

Change $\tilde{\alpha}_{lh}=\alpha_{l-I,j}r^0$, $\tilde{\beta}_{lh}=\beta_{l-Ij}r_t^2$, where l and h back to i and j to avoid too much notation,

$$g_t(0,\overline{x}_t^0,x_t)=\frac{a}{c^{(1-c)}}\sum_{k=1}^{K}\sum_{i=1}^{I}(\tilde{\alpha}_{ij}\overline{x}_t^0+\tilde{\beta}_{ij}x_t)^{1-c}\Pr\left\{(r_t^k,\omega_i)\right\}, \quad \tilde{q}_j \le \frac{x_t}{\overline{x}_t^0} \le \tilde{q}_{j+1}. \tag{30}$$

From Theorem 1, we obtain

$$V_t(\overline{x}_t^0,x_t)=\begin{cases} g_t(0,y_t^{0+},y_t^+), & \dfrac{x_t}{\overline{x}_t^0}<a_t, \\ g_t(0,\overline{x}_t^0,x_t), & a_t \le \dfrac{x_t}{\overline{x}_t^0} \le b_t, \\ g_t(0,y_t^{0-},y_t^-), & \dfrac{x_t}{\overline{x}_t^0}>b_t. \end{cases} \tag{31}$$

Assume that $\tilde{q}_{j1}<a_t \le \cdots < \tilde{q}_{j2} \le b_t < \cdots$. Define

$$\overline{q}_0=0,\overline{q}_1=a_t,\overline{q}_2=\overline{q}_{j1},\cdots,\overline{q}_{j2+j1+2}=b_t,\overline{q}_{j2+j1+3}=\infty,$$

and

$$\overline{\alpha}_{i0}=\frac{\tilde{\alpha}_{i,j1}+a_t\tilde{\beta}_{i,j1}}{1+(1+\theta)a_t},\ \ \overline{\beta}_{i0}=(1+\theta)\overline{\alpha}_{i0};$$

$$\overline{\alpha}_{ij}=\tilde{\alpha}_{i,j1+j-1},\ \ \overline{\beta}_{ij}=\tilde{\beta}_{i,j1+j-1},\ \ j=1,2,\cdots,j2-j1+1;$$

$$\overline{\alpha}_{i,j2-j1+2}=\frac{\tilde{\alpha}_{i,j2}+a_t\tilde{\beta}_{i,j2}}{1+(1-\theta)a_t},\ \ \overline{\beta}_{i,j2-j1+2}=(1-\theta)\overline{\alpha}_{i,j2-j1+2}.$$

Hence,

$$V_t(\overline{x}_t^0,x_t)=\frac{a}{c^{(1-c)}}\sum_{k=1}^{K}\sum_{i=1}^{I}(\overline{\alpha}_{ij}\overline{x}_t^0+\overline{\beta}_{ij}x_t)^{(1-c)}\Pr\{(r_t^k,\omega_i)\}, \quad \overline{q}_j \le \frac{x_t}{\overline{x}_t^0} \le \overline{q}_{j+1}, \quad j=0,1,\cdots. \tag{32}$$

Based on the above discussion, the problem (7) can be solved by the following procedures:
First, choose the proper distribution of the r_t and construct the scenario tree to determine the scenario's paths.
Secondly, use (28) and (29) to calculate the boundaries of the no-transaction region.
Finally, determine whether the portfolio lies in the no-transaction region. If it is, the portfolio is the optimal strategy; if not, take the optimal strategies as (19) and (20) shown.

4. NUMERICAL EXAMPLE

The model and the solution procedure presented in section 3 will be illustrated in this section by the numerical examples. We assume $T=5$ and consider the case in which the rate of return for the risky asset in each period is dependent of t and has only two states u and d. The Boyle and Lin [13] parameterization for u and d is used in this section, via, $u=e^{\sigma\sqrt{h}}$, $d=e^{-\sigma\sqrt{h}}$, $r^0=e^{\delta h}$ that $\sigma=1$, $h=0.25$, $\delta=0.05$.

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Thus $r^0 = 1.0126$ and the scenario data for the risky asset's return is shown in table 1.

Table 1: The risky asset's return

i th Scenario	r_1	r_2	r_3	r_4
1	1.0205	1.6825	2.7739	4.5733
2	1.0205	1.6825	2.7739	1.6824
3	1.0205	1.6825	1.0204	1.6823
4	1.0205	1.6825	1.0204	0.6189
5	1.0205	0.6189	1.0204	1.6823
6	1.0205	0.6189	1.0204	0.6189
7	1.0205	0.6189	0.3754	0.6189
8	1.0205	0.6189	0.3754	0.2277

Suppose that $a = 1$, $b = 5$, $c = 2$, we can calculate the boundaries of no-transaction region recursively backwards from the last period by using the procedures in the last section. Table 2 shows the values, when $\theta = 0.001$ and $\theta = 0.01$, respectively.

Table 2: The no-transaction bounds

	$\boldsymbol{\theta} = 0.001$	$\boldsymbol{\theta} = 0.01$
$a1$	0.559342	0.405368
$b1$	0.795427	1.07235
$a2$	0.587316	0.35647
$b2$	0.774512	1.5798
$a3$	0.526971	0.158935
$b3$	0.817125	2.547631
$a4$	0.501839	0.083563
$b4$	0.835976	2.963258

As shown in table 2, the transaction costs have a dramatic impact on the no transaction region. When the transaction costs increase, the no-transaction region has become wider. If we set the initial proportion of the risky asset is 0.05 and the initial proportion of the riskless asset is 0.95, then the investor should buy the risky asset to reach the boundary $a_1 = 0.559342$. Allowing for transaction costs, the amount of the risky asset to be purchased at time 1 is 0.5276, which is larger than the one using the power utility function as the terminal utility function in [13]. It shows that the power utility function is more risk aversion than the HARA utility function, when the two type of function have the same parameter c.

To interpret the role of the boundaries of no-transaction region, we assume that the initial wealth is 1000 dollars, of which initial proportion of risky asset is 10% and that of riskless asset is 90%. Optimal investment decisions for problem (7) are shown in Table 3 and Table 4. For each entry in these tables, the first number γ_t represents $\frac{x_t}{x_t^0 + bc(1/r^0)^{T-t}}$. According to Theorem 2, it should be compared with a_t and b_t in Table 2, then we can determine how to calculate u_t. The second number stand for the amount of risky asset one should buy or sell, while the third and forth numbers give the amount of riskless asset and risky asset one should hold, respectively. Comparing the results in table 3 with the one in table 4, we can see that the investors facing the higher transaction costs will behave more risk aversion.

Table 5 gives the optimal investment strategies in each period with different transaction costs. It shows that the investors will change his strategies according to his forecasting of the rate of the risky asset's return. Take the first scenario as an example, the scenario shows the up-up tendency while the proportion of the risky asset increases. Anyway, the above results show the efficiency of the method we proposed in this paper.

5. CONCLUSION

Multi-period investment problems with HARR utility function and proportional transaction costs are investigated in this paper. We have developed the analytical expressions for the indirect utility function and the boundary of the no-transaction region and given the optimal strategy of the investment problem. From the analysis, we can see that the results are independent with the time spacing, thus our researches are also valid for unequal time spacing. The numerical example indicates the efficiency of the method.

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Table 3: Numerical results for $\theta = 0.001$

		1	*2*	*3*	*4*	*5*
1	γ_t	0.1099	0.5554	0.9615	1.4272	-
	u_t	262.0647	13.3531	-53.2438	-283.9864	-
	x_t^0	900	655.4606	660.0601	871.5386	1443
	x_t	100	369.487	644.1284	1258.1	3270.6
2	γ_t	0.1099	0.5554	0.9615	1.4272	-
	u_t	262.0647	13.3531	-53.2438	-283.9864	-
	x_t^0	900	655.4606	660.0601	871.5386	1443
	x_t	100	369.487	644.1284	1258.1	1203.2
3	γ_t	0.1099	0.5554	0.9615	0.525	-
	u_t	262.0647	13.3531	-53.2438	0	-
	x_t^0	900	655.4606	660.0601	871.5386	892.646
	x_t	100	369.487	644.1284	426.8134	778.591
4	γ_t	0.1099	0.5554	0.9615	0.525	-
	u_t	262.0647	13.3531	-53.2438	0	-
	x_t^0	900	655.4606	660.0601	871.5386	892.646
	x_t	100	369.487	644.1284	426.8134	286.4352
5	γ_t	0.1099	0.5554	0.3537	0.5223	-
	u_t	262.0647	13.3531	76.0047	0	-
	x_t^0	900	655.4606	660.0601	601.3376	619.0405
	x_t	100	369.487	2393977	319.3284	537.2062
6	γ_t	0.1099	0.5554	0.3537	0.5223	-
	u_t	262.0647	13.3531	76.0047	0	-
	x_t^0	900	655.4606	660.0601	601.3376	619.0405
	x_t	100	369.487	239397	319.3284	197.6323
7	γ_t	0.1099	0.5554	0.3537	0.1922	-
	u_t	262.0647	13.3531	76.0047	126.0125	-
	x_t^0	900	655.4606	660.0601	601.3376	491.3126
	x_t	100	369.487	239397	117.4793	150.6971
8	γ_t	0.1099	0.5554	0.3537	0.1922	-
	u_t	262.0647	13.3531	76.0047	126.0125	-
	x_t^0	900	655.4606	660.0601	601.3376	491.3126
	x_t	100	369.487	239397	117.4793	55.4431

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Table 4: Numerical results for $\theta = 0.01$

		1	2	3	4	5
1	γ_t	0.1099	0.4031	0.661	1.7875	-
	u_t	190.6707	0	0	0	-
	x_t^0	900	726.0888	745.1131	764.5015	784.2602
	x_t	100	296.6295	499.0791	1384.4	6331.3
2	γ_t	0.1099	0.4031	0.661	1.7875	-
	u_t	190.6707	0	0	0	-
	x_t^0	900	726.0888	745.1131	764.5015	784.2602
	x_t	100	296.6295	499.0791	1384.4	2329.1
3	γ_t	0.1099	0.4031	0.661	1.7875	-
	u_t	190.6707	0	0	0	-
	x_t^0	900	726.0888	745.1131	764.5015	784.2602
	x_t	100	296.6295	499.0791	509.2603	856.7286
4	γ_t	0.1099	0.4031	0.661	0.6575	-
	u_t	190.6707	0	0	0	-
	x_t^0	900	726.0888	745.1131	764.5015	784.2602
	x_t	100	296.6295	499.0791	509.2603	315.1812
5	γ_t	0.1099	0.4031	0.2432	0.2419	-
	u_t	190.6707	0	0	0	-
	x_t^0	900	726.0888	745.1131	764.5015	784.2602
	x_t	100	296.6295	183.584	187.3291	315.1437
6	γ_t	0.1099	0.4031	0.2432	0.2419	-
	u_t	190.6707	0	0	0	-
	x_t^0	900	726.0888	745.1131	764.5015	784.2602
	x_t	100	296.6295	183.584	187.3291	115.938
7	γ_t	0.1099	0.4031	0.2432	0.089	-
	u_t	190.6707	0	0	0	-
	x_t^0	900	726.0888	745.1131	764.5015	784.2602
	x_t	100	296.6295	183.584	68.9174	42.653
8	γ_t	0.1099	0.4031	0.2432	0.089	-
	u_t	190.6707	0	0	0	-
	x_t^0	900	726.0888	745.1131	764.5015	784.2602
	x_t	100	296.6295	183.584	68.9174	15.6925

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Table 5: Proportion of riskless asset and risky asset

Scenario	*Time*	$\theta = 0.001$		$\theta = 0.01$	
		x_t^0	x_t	x_t^0	x_t
1	1	0.9000	0.1000	0.9000	0.1000
	2	0.6395	0.3605	0.7100	0.2900
	3	0.5061	0.4939	0.5989	0.4011
	4	0.4092	0.5908	0.3558	0.6442
	5	0.3061	0.6939	0.1102	0.8898
2	1	0.9000	0.1000	0.9000	0.1000
	2	0.6395	0.3605	0.7100	0.2900
	3	0.5061	0.4939	0.5989	0.4011
	4	0.4092	0.5908	0.3558	0.6442
	5	0.5453	0.4547	0.2519	0.7481
3	1	0.9000	0.1000	0.9000	0.1000
	2	0.6395	0.3605	0.7100	0.2900
	3	0.5061	0.4939	0.5989	0.4011
	4	0.6713	0.3287	0.6002	0.3998
	5	0.5341	0.4659	0.4779	0.5221
4	1	0.9000	0.1000	0.9000	0.1000
	2	0.6395	0.3605	0.7100	0.2900
	3	0.5061	0.4939	0.5989	0.4011
	4	0.6712	0.3287	0.6002	0.3998
	5	0.7571	0.2429	0.7133	0.2867
5	1	0.9000	0.1000	0.9000	0.1000
	2	0.6395	0.3605	0.7100	0.2900
	3	0.0027	0.9973	0.8023	0.1977
	4	0.6532	0.3468	0.8032	0.1968
	5	0.5354	0.4646	0.7134	0.2866
6	1	0.9000	0.1000	0.9000	0.1000
	2	0.6395	0.3605	0.7100	0.29
	3	0.0027	0.9973	0.8023	0.1977
	4	0.6532	0.3468	0.8032	0.1968
	5	0.7580	0.2420	0.8712	0.1288
7	1	0.9000	0.1000	0.9000	0.1000
	2	0.6395	0.3605	0.7100	0.2900
	3	0.0027	0.9973	0.8023	0.1977
	4	0.8366	0.1634	0.9173	0.0827
	5	0.7653	0.2347	0.9484	0.0516
8	1	0.9000	0.1000	0.9000	0.1000
	2	0.6395	0.3605	0.7100	0.2900
	3	0.0027	0.9973	0.8023	0.1977
	4	0.8366	0.1634	0.91731	0.0827
	5	0.8986	0.1014	0.9804	0.0196

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