# Fuzzy ideals of Dual QS-algebras

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**Abstract :**

The aim of this paper is introduce the notion of fuzzy KUS-ideal in KUS-algebra , several theorems, properties are stated and proved. The fuzzy relations on KUS-algebras are also studied.

**Keywords**: *KUS-algebra, fuzzy KUS-sub-algebra, fuzzy KUS-ideal, homomorphisms of KUS-algebras, image and pre-image of fuzzy KUS-ideals,*

### 1. Introduction

The concept of fuzzy subset was introduced by L.A. Zadeh in [6], and was used afterwards by many authors in various branches of mathematics. Particularly in the area of fuzzy topology . Much research has been carried out since . In 1966 [9], Y. Imai and K. Is´eki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [4],[5]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Xi [7] applied the concept of fuzzy subset to BCK-algebras and gave some of its properties . J. Neggers , S.S. Ahn and H.S. Kim [3] introduced a Q-algebra, which is a generalization of BCI/BCK-algebras and generalized some theorems discussed in BCI-algebra . Moreover, Ahn and Kim [8] introduced the notion of QS-algebra which is a paper subclass of Q-algebra. In 2013 [2], introduced a new notion called KUS-algebra, which is dual for QS-algebra and investigated severed basic properties which are related to KUS-ideal . In this paper , we introduce the notion of fuzzy KUS-ideals in KUS-algebras and then we investigate several basic properties which are related to fuzzy KUS-ideals. We describe how to deal with the homomorphism of image and inverse image of fuzzy KUS-ideals of KUS-algebras.

## 2. Preliminaries

**Definition 2.1([2]).** Let $(X; *,0)$ be an algebra with a single binary operation $(*)$ . X is called a KUS-algebra if it satisfies the following identities: for any $x, y, z \in X$ ,

$(kus_1) : (z * y) * (z * x) = y * x$ ,
$(kus_2) : 0 * x = x$ ,
$(kus_3) : x * x = 0$ ,
$(kus_4) : x * (y * z) = y * (x * z)$ .

In X we can define a binary relation $(\leq)$ by: $x \leq y$ if and only if $y * x = 0$ .

In what follows, let $(X; *,0)$ denote a KUS-algebra unless otherwise specified. For brevity we also call X a KUS-algebra.

**Lemma 2.2 ([2]).** In any KUS-algebra $(X; *,0)$ , the following properties hold: for all $x, y, z \in X$;

a) $x * y = 0$ and $y * x = 0$ imply $x = y$,
b) $y * [(y * z) * z] = 0$,
c) $(0 * x) * (y * x) = y * 0$ ,
d) $(x * y) * 0 = y * x$ ,

**Theorem 2.3([2]).** Any KUS-algebra is equivalent to the dual QS- algebra.

**Definition 2.4([2]).** Let X be a KUS-algebra and let S be a nonempty subset of X. S is called a KUS-sub-algebra of X if $x * y \in S$ whenever $x \in S$ and $y \in S$.

**Definition 2.5 ([2]).** A nonempty subset I of a KUS-algebra X is called a KUS-ideal of X if it satisfies: for x , y, z ∈ X,

($Ikus_1$) (0 ∈ I) ,

($Ikus_2$) (z ∗ y)∈ I and (y ∗ x)∈ I imply (z ∗ x)∈ I.

**Example 2.6 .** Let X ={0 ,a, b, c} in which ( ∗ ) is defined by the following table:

| * | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

Then (X; ∗ ,0) is a KUS-algebra . It is easy to show that $I_1$ ={0,a}, $I_2$ ={0,b}, $I_3$={0,c}, and $I_4$ = {0, a, b, c} are KUS-ideals of X .

**Proposition 2.7([2]).** Every KUS-ideal of KUS-algebra X is a KUS-sub-algebra.

## 3. Fuzzy KUS-ideals and Homomorphism of KUS-algebras

In this section , we will discuss a new notion called fuzzy KUS-ideals of KUS-algebras and study several basic properties which are related to fuzzy KUS-ideals .

**Definition 3.1([6]).** Let (X; ∗ ,0) be a nonempty set, a fuzzy subset μ in X is a function μ: X → [0,1].

**Definition 3.2**. Let (X; ∗ ,0) be a KUS-algebra , a fuzzy subset μ in X is called a fuzzy KUS-sub-algebra of X if for all x , y ∈ X, μ (x ∗ y) ≥ min {μ (x), μ (y)} .

**Definition 3.3**. Let (X; ∗ ,0) be a KUS-algebra , a fuzzy subset μ in X is called a fuzzy KUS-ideal of X if it satisfies the following conditions: , for all x , y, z ∈ X ,

($Fkus_1$) μ (0) ≥ μ (x) ,

($Fkus_2$) μ (z ∗ x) ≥ min {μ (z ∗ y), μ (y ∗ x)}.

**Example 3.4.**

1) Let X ={0, 1, 2, 3} in which ( ∗ ) is defined by the following table:

| ∗ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

Then ( X ; ∗ , 0) is a KUS-algebra . Define a fuzzy subset μ : X→ [0,1] by

$$\mu(x) = \begin{cases} 0.7 & \text{if } x \in \{0,1\} \\ 0.3 & \text{otherwise} \end{cases}$$

$I_1$ = {0, 1} is a KUS-ideal of X. Routine calculation gives that μ is a fuzzy KUS-ideal of KUS-algebras X.

2) Consider X = {0, a, b, c, d} with( ∗ ) defined by the table :

| ∗ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

Then (X ; ∗ ,0) is a KUS-algebra. Define a fuzzy subset μ: X → [0,1]such that μ(0) = $t_1$ , μ(a) = μ(b) = μ(c) = μ(d) = $t_2$ ,where $t_1$, $t_2$ ∈[0, 1] and $t_1 > t_2$.

Routine calculation gives that μ is a fuzzy KUS-ideal of KUS- algebra X .

**Definition 3.5 ([6]).** Let X be a nonempty set and μ be a fuzzy subset in X, for t ∈ [0,1] , the set $\mu_t$ ={ x ∈ X | μ(x) ≥ t} is called a level subset of μ .

**Theorem 3.6.** Let μ be a fuzzy KUS-ideal in KUS-algebra X. μ is a fuzzy KUS-ideal of X if and only if , for every t ∈ [0,1] , $\mu_t$ is either empty or a KUS-ideal of X .

**Proof:** Assume that μ is a fuzzy KUS-ideal of X , by ($Fkus_1$) , we have $\mu(0) \geq \mu(x)$ for all $x \in X$ therefore $\mu(0) \geq \mu(x) \geq t$ for $x \in \mu_t$ and so $0 \in \mu_t$ . Let $x, y, z \in X$ be such that $(z * y) \in \mu_t$ and $(y * x) \in \mu_t$ , then $\mu(z * y) \geq t$ and $\mu(y * x) \geq t$ , since μ is a fuzzy KUS-ideal, it follows that $\mu(z * x) \geq \min\{\mu(z * y), \mu(y * x)\} \geq t$ and we have that $x * z \in \mu_t$ . Hence $\mu_t$ is a KUS-ideal of X .

Conversely , we only need to show that ($Fkus_1$) and ($Fkus_2$) are true . If ($Fkus_1$) is false , then there exist $x` \in X$ such that $\mu(0) < \mu(x`)$. If we take $t` = (\mu(x`) + \mu(0))/2$ , then $\mu(0) < t`$ and $0 \leq t` < \mu(x`) \leq 1$ , then $x` \in \mu$ and $\mu \neq \varnothing$ . As $\mu_{t`}$ is a KUS-ideal of X , we have $0 \in \mu_{t`}$ and so $\mu(0) \geq t`$. This is a contradiction . Now , assume ($Fkus_2$) is not true ,then there exist $x`, y`, z` \in X$ such that , $\mu(z` * x`) < \min\{\mu(z` * y`), \mu(y` * x`)\}$. Putting $t` = (\mu(z` * x`) + \min\{\mu(z` * y`), \mu(y` * x`)\})/2$ , then $\mu(x` * z`) < t`$ and $0 \leq t` < \min\{\mu(z` * x`), \mu(y` * x`)\} \leq 1$, hence $\mu(z` * y`)) > t`$ and $\mu(y` * x`) > t`$, which imply that $(z` * y`) \in \mu_{t`}$ and $(y` * x`) \in \mu_{t`}$ .

Since $\mu_{t`}$ is a KUS-ideal ,it follows that $(x` * z`) \in \mu_{t`}$ and that $\mu(x` * z`) \geq t`$, this is also a contradiction . Hence μ is a fuzzy KUS-ideal of X . ⌂

**Corollary 3.7.** Let μ be a fuzzy subset in KUS-algebra X . If μ is a fuzzy KUS-ideal, then for every $t \in Im(\mu)$ , $\mu_t$ is a KUS- ideal of X when $\mu_t \neq \varnothing$.

**Theorem 3.8.** Let μ be a fuzzy subset in KUS- algebra X . If μ is a fuzzy KUS-sub-algebra of X if and only if , for every $t \in [0,1]$ , $\mu_t$ is either empty or a KUS-sub-algebra of X.

**Proof:** Assume that μ is a fuzzy KUS-sub-algebra of X ,let $x, y \in X$ be such that $x \in \mu_t$ and $y \in \mu_t$ , then $\mu(x) \geq t$ and $\mu(y) \geq t$ . Since μ is a fuzzy KUS-sub-algebra, it follows that $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq t$ and that $(x * y) \in \mu_t$ .

Hence $\mu_t$ is a KUS-sub-algebra of X .

Conversely, assume $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ is not true ,then there exist $x`$ and $y` \in X$ such that , $\mu(x` * y`) < \min\{\mu(x`), \mu(y`)\}$. Putting $t` = (\mu(x` * y`) + \min\{\mu(x`), \mu(y`)\})/2$ , then $\mu(x`) < t`$ and $0 \leq t` < \min\{\mu(x`), \mu(y`)\} \leq 1$, hence $\mu(x`) > t`$ and $\mu(y`) > t`$, which imply that $x` \in \mu_{t`}$ and $y` \in \mu_{t`}$ , since $\mu_{t`}$ is a KUS-sub-algebra ,it follows that $x` * y` \in \mu_{t`}$ and that $\mu(x` * y`) \geq t`$, this is also a contradiction . Hence μ is a fuzzy KUS-sub-algebra of X . ⌂

**Proposition 3.9.** Every fuzzy KUS-ideal of KUS-algebra X is a fuzzy KUS-sub-algebra of X **.**

**Proof:** Since μ is fuzzy KUS-ideal of a KUS-algebra X , then by theorem (3.6) , for every $t \in [0,1]$ , $\mu_t$ is either empty or a KUS-ideal of X . By proposition(2.7), for every $t \in [0,1]$, $\mu_t$ is either empty or a KUS-sub-algebra of X .Hence μ is a fuzzy KUS-sub-algebra of KUS-algebra X by theorem (3.8). ⌂

**Definition 3.10 ([1]).** Let $(X ; *, 0)$ and $(Y; *`, 0`)$ be nonempty sets. The mapping $f : (X; *, 0) \rightarrow (Y; *`, 0`)$ is called a homomorphism if it satisfies: $f(x * y) = f(x) *` f(y)$, for all $x, y \in X$. The set $\{x \in X \mid f(x) = 0'\}$ is called the Kernel of $f$ denoted by Ker $f$ .

**Definition 3. 11 ([1]).** Let $f$ : (X;*,0) →(Y;*',0') be a mapping nonempty sets X and Y respectively. If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by:

$$f(\mu)(y)=\begin{cases}\sup\{\mu(x): x\in f^{-1}(y)\} & \text{if } f^{-1}(y)=\{x\in X, f(x)=y\}\neq\phi\\ 0 & \text{otherwise}\end{cases}$$

is said to be the image of μ under $f$.

Similarly if β is a fuzzy subset of Y, then the fuzzy subset μ = (β o $f$ ) in X ( i.e the fuzzy subset defined by μ (x) = β ( $f$ (x)) for all x ∈ X) is called the pre-image of β under $f$ .

**Theorem 3.12**. An into homomorphic pre-image of a fuzzy KUS-ideal is also a fuzzy KUS-ideal .

**Proof:** Let $f$ : (X; ∗,0) →(Y; ∗',0') be an into homomorphism of KUS-algebras, β a fuzzy KUS-ideal of Y and μ the pre-image of β under f , then β ( $f$ (x)) = μ (x) , for all x ∈ X . Since $f$ (x) ∈ Y and β is a fuzzy KUS-ideal of Y, it follows that

β(0') ≥ β( $f$ (x)) = μ(x) , for every x ∈ X, where 0' is the zero element of Y.

But β(0') = β( $f$ (0)) = μ(0) and so μ(0) ≥ μ(x) for x ∈ X.

Now let x, y, z ∈ X , then we get

μ (z ∗ x) = β ( $f$ (z ∗ x)) = β( $f$ (z) ∗` $f$ (x))

≥ min {β ( $f$ (z) ∗` $f$ (y)), β( $f$ (y) ∗` $f$ (x))}

= min {β ( $f$ (z ∗ y)),β( $f$ (y ∗ x)) }

= min{μ(z∗y)), μ (y∗x)}

i.e., μ (z∗x) ≥ min{μ(z∗y)), μ (y∗x)}, for all x, y, z ∈ X. ⌂

**Definition 3.13 ([1]).** A fuzzy subset μ of a set X has sup property if for any subset T of X , there exist $t_0$ ∈T such that μ($t_0$) = sup {μ(t)| t∈T}.

**Theorem 3.14.** Let $f$ : (X; ∗,0) →(Y; ∗',0') be a homomorphism between KUS-algebras X and Y respectively. For every fuzzy KUS-ideal μ in X with sup property, $f$ (μ) is a fuzzy KUS-ideal of Y.

**Proof:** By definition β(y') = $f$ (μ)(y') := sup{μ(x)| $x\in f^{-1}(y^{\backslash})$ },for all y'∈ Y (sup ∅ = 0).

We have to prove that β(z'∗' x') ≥ min {β(z'∗' y'),β(y'∗' x')}, for all x', y', z' ∈ Y.

**(I)** Let $f$ : (X; ∗,0) →(Y; ∗',0') be a onto homomorphism of KUS-algebras, μ is a fuzzy KUS-ideal of X with sup property and β the image of μ under $f$ . Since μ is a fuzzy KUS-ideal of X , we have μ(0) ≥ μ(x) for all x ∈ X. Note that 0 ∈ $f^{-1}$ (0'), where 0 and 0' are the zero elements of X and Y respectively. Thus

$\beta(0^{\backslash})=\sup_{t\in f^{-1}(x^{\backslash})}\mu(t)=\mu(0)\geq\mu(x)$ = μ(0) ≥ μ(x)

for all x ∈ X, which implies that

$\beta(0^{\backslash})\geq\sup_{t\in f^{-1}(x^{\backslash})}\mu(t)=\beta(x^{\backslash})$ for any $x^{\backslash}\in Y$

For any x', y', z' ∈ Y , let $x_0\in f^{-1}(x^{\backslash})$ , $y_0\in f^{-1}(y^{\backslash})$, $z_0\in f^{-1}(z^{\backslash})$ be such that:

$$\mu(z_0*y_0)=\beta[f(z_0*y_0)]=\beta[f(z^{\backslash}*y^{\backslash})]=\sup_{z_0*y_0\in f^{-1}(z^{\backslash}*y^{\backslash})}\mu(z_0*y_0)$$

$$\mu(y_0*x_0)=\beta[f(y_0*x_0)]=\beta[f(y^{\backslash}*x^{\backslash})]=\sup_{y_0*x_0\in f^{-1}(y^{\backslash}*x^{\backslash})}\mu(y_0*x_0)$$

, then

$$\beta(z^{\backslash}*x^{\backslash})=\sup\mu(t)=\mu(z_0*x_0)$$

$$\geq\min_{t\in f^{-1}(z^{\backslash}*x^{\backslash})}\{\mu(z_0*y_0),\mu(y_0*x_0)\}$$

$$=\min\left\{\sup_{t\in f^{-1}(z^{\backslash}*y^{\backslash})}\mu(t)\ ,\ \sup_{t\in f^{-1}(y^{\backslash}*x^{\backslash})}\mu(t)\right\}$$

$$=\min\{\beta(z^{\backslash}*y^{\backslash}),\beta(y^{\backslash}*x^{\backslash})\}$$

Hence β is a fuzzy KUS-ideal of Y.

**(II)** If f is not onto: For every $x^\backslash \in Y$, we define $X_{x^\backslash} := f^{-1}(x^\backslash)$ .Since f is a homomorphism ,we get

$X_{z^\backslash} * X_{y^\backslash} \subset X_{z^\backslash * y^\backslash}$ *and* $X_{y^\backslash} * X_{x^\backslash} \subset X_{y^\backslash * x^\backslash}$, for all x', y', z' ∈ Y ------------- (*).

Let x' ,y' ,z' ∈ Y be arbitrarily given. If $(z^\backslash * y^\backslash) \notin \mathrm{Im}(f) = f(X)$ , then by definition $\beta(z^\backslash * y^\backslash) = 0$ . But if $(z^\backslash * y^\backslash) \notin f(X)$, i.e. , $X_{z^\backslash * y^\backslash} = \phi$ , then by (*) at least one of $z^\backslash, y^\backslash, x^\backslash \notin f(X)$ and hence β(z'*'x') ≥ 0 = min {β(z'*'y'),β(y'*'x')}. ⌂

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