Non-equidistant GRM(1,1) Generated by Accumulated Generating Operation of Reciprocal Number and its Application

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Abstract

Applying the modeling method of grey system and accumulated generating operation of reciprocal number for the problem of lower precision as well as lower adaptability in non-equidistant GM(1,1) model, a non-equidistant GRM(1,1) model generated by accumulated generating operation of reciprocal number was put forward. Based on index characteristic of grey model and the definition of integral method, the background value in non-equidistant GRM(1,1) was researched and the discrete function with non-homogeneous exponential law was used to fit the accumulated sequence and optimum formula was given. The formula of background value of GRM(1,1) model can be used in non-equal interval and equal interval time series and has the characteristic of high precision as well as high adaptability. Example validates the practicability and reliability of the proposed model.

Keywords: *Background value, grey model GRM(1,1) generated by accumulated generating operation of reciprocal number, non-equal interval, accumulation generation operation, grey system.*

1. Introduction

The grey model is one important part of the grey theory since it is provided by Deng and it has been widely used in many fields[1]. GM(1,1) model is characteristic with the small samples and poor information and it is important for the experimental data processing and on-line monitoring [2-9]. The grey system is based on the equal interval but in fact there are much data which is in the unequal interval. So establishment of the unequal interval model is meaningful for the theory and practice. For the non-negative discrete data $x^{(0)}$ the sequence $x^{(1)}$ after AGO process is monotonic increasing. It is reasonable that the curve is monotonic increasing which is used to fit $x^{(1)}$. But if $x^{(0)}$ is monotonic decreasing that the AGO operation determines $x^{(1)}$ is monotonic increasing. So the fitting $\hat{x}^{(1)}$ model is monotonic increasing. By the IAGO process to restore the original data it will produce some unreasonable error. So for the monotonic decreasing original $x^{(0)}$ literature [10] provides inverse accumulated operation and creates the GOM(1,1) model based on the inverse accumulated operation. Literature [11] provides reciprocal operation and creates the GRM(1,1) model based on the reciprocal operation; literature [12] improved the GRM(1,1) model and established the improved CGRM(1,1) model based on the reciprocal operation. The model which is based on the reciprocal operation and inverse accumulated operation makes $x^{(1)}$ be monotonic decreasing, then uses the monotonic decreasing curve to fit $x^{(1)}$ and gets the fitting $\hat{x}^{(1)}$. After restoring the original sequence $x^{(0)}$ it will not produced the unreasonable error and it can improve the accuracy of the model. Literature [2] takes the interval as the multiplier to establish the unequal GM(1,1) model in which the difference in values of the data has no linear relation with the time and the construction of the model is not fit for the reality. Literature [3] adopt the function-transformation to decrease coefficient of standard error and put the original data into a new sequence, estimates the parameters of the model and improves the accuracy of GM(1,1) model which is complicated. Literature [4-9] provides many construction forms of the background values and establish some kinds of unequal GM(1,1) model. Literature [4-5] improves the method of background values and it adopts the homogenous index function to fit the sequence after AGO and get high accuracy. But from the form of solutions of the whitening differential equations it should be non-homogenous index form and only after IAGO process it is the homogenous form, so using the homogenous form to fit is not accurate. Literature [6] uses the non-homogenous index form to fit the sequence after AGO process and get the optimized construction of the background values and establish the unequal GM(1,1) model. Literature [7] uses the non-homogenous index function to fit the sequence after AGO process and induce the optimized construction of the background values and get unequal interval GM(1,1) model based on the optimized background values. Literature [8] adopts the optimized coefficient λ in the background values ($z^{(1)}(t_k)=\lambda x^{(1)}(t_k)+(1-\lambda)x^{(1)}(t_{k-1}),\lambda\in[0,1]$) , optimized the fix term $\beta = [\beta_1, \beta_2, \cdots, \beta_n]^{\mathbf{T}}$ in the initial condition and establish the unequal GM(1,1) model. The paper absorbs the construction method of the background values in literature [7] and takes the non-homogenous index function to get the optimized background values. Based on the initial condition of taking the first component of $\mathbf{x}^{(0)}$ as the initial condition it establishes the unequal reciprocal accumulated generating operation GRM(1,1) model. The model has high accuracy and is good for theory and practice. The examples of processing fatigue experiment show it is effective and reliable.

2. The Unequal GRM(1,1) Model Based on Reciprocal Accumulated Operation

Definition 1 Given $X^{(00)} = [x^{(00)}(t_1), \cdots, x^{(00)}(t_m)]$, if $\Delta t_i = t_i - t_{i-1} \neq const, \ i = 2, \cdots, m$, $X^{(00)}$ is the unequal sequence. Given $x^{(0)}(t_k) = \frac{1}{x^{(00)}(t_k)} (k = 1, 2, \cdots, m)$, $\mathbf{x}^{(0)} = (x^{(0)}(t_1), \cdots, x^{(0)}(t_m))$ is the reciprocal sequence of $\mathbf{x}^{(00)}$.

Definition 2 Given $X^{(1)} = [x^{(1)}(t_1), x^{(1)}(t_2), \cdots, x^{(1)}(t_n)]$, ($x^{(0)}(t_k) = \frac{1}{x^{(00)}(t_k)} (k = 1, 2, \cdots, m)$,if $x^{(1)}(t_1) = x^{(0)}(t_1)$, $x^{(1)}(t_{k+1}) = x^{(1)}(t_k) + x^{(0)}(t_{k+1}) \cdot \Delta t_{k+1}, k = 1, \cdots, m-1$, $X^{(1)}$ is called the sequence of one-order reciprocal accumulated operation.

If the original sequence is $\mathbf{X}^{(00)} = [x^{(00)}(t_1), \cdots, ^{(00)}(t_m)]$, $x^{(00)}(t_j)(j = 1, 2, \cdots m)$ is the objective value at t_j; m is the number of the data, the sequence $[x^{(0)}(t_1), x^{(0)}(t_2), \cdots, x^{(0)}(t_m)]$ is the unequal interval, namely $t_j - t_{j-1}$ is not the const.

In order to establish the model we adopt the reciprocal AGO process to form the new sequence.

$$\mathbf{X}^{(1)} = [x^{(1)}(t_1), x^{(1)}(t_2), \cdots, x^{(1)}(t_m)] \quad (1)$$

Where $x^{(1)}(t_j)(j = 1, 2, \cdots, m)$ satisfies the definition2, namely

$$x^{(1)}(t_k) = \begin{cases} \sum_{j=1}^{k} x^{(1)}(t_j)(t_j - t_{j-1}) & (k = 2, \cdots, m) \\ x^{(0)}(t_1) & (k = 1) \end{cases} \quad (2)$$

For the sequence $x^{(1)}$ after the one-order reciprocal AGO process we can establish the unequal GRM(1,1) model $\frac{dx^{(1)}}{dt} + a\zeta^{(1)} = b$ ($\zeta^{(1)}$ is the background values) and the whitening differential equations are :

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = b \quad (3)$$

The form of difference equation is:

$$x^{(0)}(t_k) + az^{(1)}(t_k) = b \quad (4)$$

Where

$$z^{(1)}(t_k) = 0.5 * (x^{(1)}(t_k) + x^{(1)}(t_{k-1}))$$

$z^{(1)}(t_k)$ is the background values in GRM(1,1) model and it is the average operation values.

If $\hat{\mathbf{a}} = [a, b]^{\mathbf{T}}$ is the parameters in unequal GRM(1,1) and by the least square method the estimated values is:

$$\hat{\mathbf{a}} = (\mathbf{B}^{\mathbf{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{Y} \quad (5)$$

Where

$$\mathbf{B} = \begin{bmatrix} -\frac{1}{2}(x^{(1)}(t_2) + x^{(1)}(t_1)) & 1 \\ -\frac{1}{2}(x^{(1)}(t_3) + x^{(1)}(t_2)) & 1 \\ \cdots & \cdots \\ -\frac{1}{2}(x^{(1)}(t_m) + x^{(1)}(t_{m-1})) & 1 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} x^{(0)}(t_2) \\ x^{(0)}(t_3) \\ \cdots \\ x^{(0)}(t_m) \end{bmatrix}$$

The time response equations of grey differential equation (4) are

$$\hat{x}^{(1)}(t_k) = \frac{\hat{b}}{\hat{a}} + (x^{(0)}(t_1) - \frac{\hat{b}}{\hat{a}})e^{-a(t_k - t_1)} (k = 1, 2, \cdots, m) \quad (6)$$

The fitting data after restoring are

$$\hat{x}^{(0)}(t_k)=\begin{cases} x^{(0)}(t_1) \quad (k=1) \\ \dfrac{(x^{(0)}(t_1)-\frac{\hat{b}}{\hat{a}})(1-e^{a\Delta t_k})}{\Delta t_k} \quad (k=2,3,\cdots,m) \end{cases} \tag{7}$$

Using the definition (1) the fitting values of the original sequence are $\hat{x}^{(00)}(t_k)(k=1,2,\cdots,m)$.

The absolute error of the fitting data is

$$q(t_k)=\hat{x}^{(00)}(t_k)-x^{(00)}(t_k) \tag{8}$$

The relative error of the fitting data is

$$e(t_k)=\frac{\hat{x}^{(00)}(t_k)-x^{(00)}(t_k)}{x_i^{(0)(00)}(t_k)}*100 \tag{9}$$

The average values of the relative error are

$$f=\frac{1}{m}\sum_{k=1}^{m}|e_i(k)| \tag{10}$$

3. The Optimized Background Values in the Unequal Interval GRM(1,1) Model

The traditional average values are to use trapezium formula to fit the area which is combined by curve $x^{(1)}(t)$ and x axis between intervals $[t_k,t_{k+1}]$. When the interval is little and the sequence is changed smoothly, the construction of background values is accurate. When the data are changed fast the construction form of the background values are not accurate and it will produce a large error. So we must research the construction of the background values. Integrating the whitening equation $\frac{dx^{(1)}}{dt}+ax^{(1)}=b$ between $[t_k,t_{k+1}]$ and obtain

$$x^{(0)}(t_k)\Delta t_k+a\int_{t_{k-1}}^{t_k}x^{(1)}dt=b\Delta t_k \tag{11}$$

Compared the grey differential equations $x^{(0)}(t_k)+az^{(1)}(t_k)=b$ with formula (11) the form of $\int_{t_{k-1}}^{t_k}x^{(1)}dt$ is more accurate as the construction form of the background values. And from formula (8) $x^{(0)}(t_k)\Delta t_k+a\int_{t_{k-1}}^{t_k}x^{(1)}dt=b\Delta t_k$ can satisfy the homogeneous index form. Given $x^{(1)}(t_k)=Ge^{-a(t_k-t_1)}+C$, the sequence after IAGO is

$$\begin{aligned} x^{(0)}(t_k)&=\frac{x^{(1)}(t_k)-x^{(1)}(t_{k-1})}{\Delta t_k} \\ &=\frac{G(1-e^{a\Delta t_i})}{\Delta t_k}e^{-a(t_k-t_1)}=g_k e^{-a(t_k-t_1)} \end{aligned} \tag{12}$$

Where

$$g_k=\frac{G(1-e^{a\Delta t_k})}{\Delta t_k}=\frac{G(1-(1+(a\Delta t_k)+\frac{(a\Delta t_k)^2}{2!}+\cdots)}{\Delta t_k}$$

When a and Δt_k is little we expand the first two polynomial of $e^{a\Delta t_i}$, then we obtain

$$g_k=\frac{G(1-e^{a\Delta t_k})}{\Delta t_k}=\frac{G(-a\Delta t_k)}{\Delta t_k}=-Ga$$

$$\frac{x^{(0)}(t_k)}{x^{(0)}(t_{k-1})}=\frac{e^{-a(t_k-t_1)}}{e^{-a(t_{k-1}-t_1)}}=e^{-a\Delta t_k}$$

So

$$a=-\frac{\ln x^{(0)}(t_k)-\ln x^{(0)}(t_{k-1})}{\Delta t_k}(k=2,3,\cdots,m) \tag{13}$$

Put formula (13) into (12), then

$$\begin{cases} g_k=\dfrac{x^{(0)}(t_k)}{e^{-a(t_k-t_1)}}=\dfrac{x^{(0)}(t_k)}{[x^{(0)}(t_k)/x^{(0)}(t_{k-1})]^{\frac{t_1-t_k}{\Delta t_k}}} \\ G=\dfrac{x^{(0)}(t_k)\Delta t_k[x^{(0)}(t_k)/x^{(0)}(t_{k-1})]^{\frac{t_1-t_k}{\Delta t_k}}}{1-\dfrac{x^{(0)}(t_{k-1})}{x^{(0)}(t_k)}} \end{cases} \tag{14}$$

From the initial condition $x^{(1)}(t_1)=Ge^{-a(t_1-t_1)}+C=G+C$ then we obtain

$$C=\hat{x}^{(0)}(t_1)-G$$

$$=\hat{x}^{(0)}(t_1)-\frac{x^{(0)}(t_k)\Delta t_k[x^{(0)}(t_k)/x^{(0)}(t_{k-1})]^{\frac{t_1-t_k}{\Delta t_k}}}{1-\frac{x^{(0)}(t_{k-1})}{x^{(0)}(t_k)}} \quad (15)$$

Put formula (15) and (17) into the construction form $\int_{t_{k-1}}^{t_k} x^{(1)} dt$ of background values,

$$z^{(1)}(t_k)=\int_{t_{k-1}}^{t_k} x^{(1)} dt=-\frac{\Delta t_k x^{(0)}(t_k)}{a}+C\Delta t_k$$
$$=\frac{(\Delta t_k)^2 x^{(0)}(t_k)}{\ln x^{(0)}(t_k)-\ln x^{(0)}(t_{k-1})}+\hat{x}^{(0)}(t_1)\Delta t_k \quad (16)$$
$$-\frac{x^{(0)}(t_k)(\Delta t_k)^2[x^{(0)}(t_k)/x^{(0)}(t_{k-1})]^{\frac{t_1-t_k}{\Delta t_k}}}{1-\frac{x^{(0)}(t_{k-1})}{x^{(0)}(t_k)}}$$

By the least square method in the whitening differential equations $x^{(0)}(t_k)\Delta t_k + a\int_{t_{k-1}}^{t_k} x^{(1)} dt = b\Delta t_k$ the estimated parameters are:

$$\hat{\mathbf{a}} = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Y} \quad (17)$$

Where

$$\mathbf{B}=\begin{bmatrix} -z^{(1)}(t_2) & \Delta t_2 \\ -z^{(1)}(t_2) & \Delta t_3 \\ \cdots & \cdots \\ -z^{(1)}(t_m) & \Delta t_m \end{bmatrix}, \quad \mathbf{Y}=\begin{bmatrix} x^{(0)}(t_2)\Delta t_2 \\ x^{(0)}(t_3)\Delta t_3 \\ \cdots \\ x^{(0)}(t_m)\Delta t_m \end{bmatrix}$$

Put formula (19) into (8)-(12) and obtain GRM(1,1) model which is unequal model with the fitting, predictive and error values and then check the model[13-15].

3. Examples

The fatigue strength is changed with the temperature and the data are as table 1. Obviously it is the unequal interval sequence. The paper adopts the data in literature [2,3] and uses the method in the paper to establish a model. And it can be obtained.

a= -0.0009873,b=0.0017674.

$$\hat{x}^{(1)}(t_k) = 3.5821e^{0.0009873(t-100)} - 1.7901$$

Tab.1 Relation of Ti alloy fatigue strength with temperature (Mpa)

$T/°C(t_k)$	100	130	170	210	240
$\sigma_{-1}(x^{(0)}(t_k))$	560	557.54	536.10	516.10	505.60
$T/°C(t_k)$	270	310	340	380	
$\sigma_{-1}(x^{(0)}(t_k))$	486.1	467.4	453.8	436.4	

The fitting data of the original data:

$\hat{x}^{(00)}(t_k) =$ [560,556.9099, 537.9791, 517.1473, 499.5965, 485.016, 468.529, 452.6282, 437.2422, 424.502]

The absolute error of the fitting data:

$q(t_k) =$ [0, 0.63008,-1.8791,-1.0473, 6.0035, 1.084, -1.129, 1.1718, -0.84222]

The relative error of the fitting data:

$e(t_k) =$ [0, 0.11301, -0.35051, -0.20293, 1.1874, 0.223, -0.24156, 0.25821, -0.19299]

The average relative error of the fitting data is 0.30773%. The literature [2] adopted a pre-processing of $t=(T-50)/50$, $x^{(0)}=(\sigma_{-1}-400)/50$ to the original sequence. The largest relative error is 4.86% and the average relative error is 3.19%. In literature [3] it uses the function-transformation method and the average relative error is 0.6587%. In literature [5] it uses the homogenous function to fit accumulated generating operation sequence and the relative error is 0.9765%. So the method in the paper is effective and scientific. The combination forecast model [16] can be built with the proposed model instead of traditional GM(1,1) model and the model parameters can also be found the optimum method [17-18].

4. CONCLUSIONS

Based on the initial condition of taking the first component of $\mathbf{x}^{(0)}$ as the initial condition it establishes the unequal reciprocal accumulated generating operation GRM(1,1) model and compiles the MATLAB program. It research the construction form of background values in the unequal GM(1,1) model and according to the index law and integral characteristic it used the non-homogeneous index function to fit the sequence of accumulated generated operation, provided unequal GRM(1,1) model based on reconstruction of background values and gives a form of the background values. The model is fit for equal and unequal interval. It model has high accuracy and is good for theory and practice.

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