Shape Analysis of Planar Trigonometric Bézier Curves with Two Shape Parameters

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Abstract
The shape features of planar trigonometric Bézier curves with two shape parameters are analyzed. The necessary and sufficient conditions are derived for these curves having one or two inflection points, a loop or a cusp, or be locally or globally convex. All conditions are completely characterized by the relative position of the control polygon's side vectors. Furthermore we discussed the influences of shape parameters on the conditions and the ability for adjusting the shape of the curve.
Keywords: *Trigonometric Bézier curve, Loop, Cusp, Inflection point, Local convexity, Global convexity.*

1. Introduction

Recently, trigonometric splines and polynomials with shape parameters have gained very much interest within CAGD, in particular curve design [1–4]. The paper [1-2] described the trigonometric polynomial curves with a global shape parameter. The paper [3] provided the C^2 continuous quadratic trigonometric polynomial curves with a local shape parameter. The cubic trigonometric Bézier curves with two shape parameters were discussed in [4]. Similar such studies using trigonometric splines can be found in [5-6]. The applications of trigonometric splines have led to the introduction of various types of trigonometric spline for CAGD purpose [7–10].

For many applications in geometric modeling it is often necessary to determine if the curve has singularities (loops or cusps), or inflection points, or is globally convex. This topic (also known as shape classification or geometric characterization of a curve in CAGD) has been studied before from different points of view [11–25]. For the case of general parametric curves the reader can see [11-12]. For planar cubic parametric curves some useful results can be found in [13–18]. For the rational case one can refer to [19–21]. For C-curves a classical shape diagram (similar to those in [13-14]) was obtained in [22]. However in the papers [11–22], the difference between global and local convexity was not referred to. In [23] a necessary and sufficient condition for global convexity of planar curves was presented. In [24], the author did not only investigate inflection points and singularities but also the global and local convexity of the planar cubic H-Bézier curves. A shape diagram (like that in [24]) of cubic trigonometric Bézier curves with a shape parameter was obtained in [25].

In this work, the shape features of planar trigonometric Bézier curves with two shape parameters are analyzed, by using the method based on the theory of envelope and topological mapping [18]. We give the conditions on the convexity and the existence of inflection points and singularities. Its use enables us to place control points and especially to choose shape parameters so that the resulting curves have not the undesirable features such as cusps and loops. The results are summarized in a shape diagram analogous to those in [24-25]. The influences of shape parameters on the shape features of the curve are also discussed.

The rest of this paper is organized as follows. In Section 2, we introduce the construction of the cubic trigonometric Bézier curves with two shape parameters (T-Bézier curve, for short). In Section 3, the cusps, inflection points, loops and convexity of the planar T-Bézier curve are discussed. In Section 4, the influences of shape parameters on the shape diagram and the ability for adjusting the shape of the curve are analyzed. We close in Section 5 with a brief summary of our work.

2. Cubic Trigonometric Bézier Curve with Two Shape Parameters

In [4] a cubic trigonometric Bézier curve with two shape parameters was defined in the following way:

Definition 2.1 Given control points $\boldsymbol{P}_i (i = 0,1,2,3)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the cubic trigonometric Bézier curve with two shape parameters is defined by

$$\boldsymbol{r}(t) = \sum_{i=0}^{3} \boldsymbol{P}_i b_i(t), \quad t \in [0,1], \tag{1}$$

where

$$\begin{cases} b_0(t) = [1 - s(t)]^2 [1 - \lambda s(t)], \\ b_1(t) = s(t)[1 - s(t)][2 + \lambda - \lambda s(t)], \\ b_2(t) = c(t)[1 - c(t)][2 + \mu - \mu c(t)], \\ b_3(t) = [1 - c(t)]^2 [1 - \mu c(t)], \end{cases} \tag{2}$$

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in which $s(t) := \sin(\pi t/2)$, $c(t) := \cos(\pi t/2)$, and λ, $\mu \in [-2,1]$ are shape parameters. It is called T-Bézier curve, for short.

In the following sections we discuss the shape features of the planar T-Bézier curves. Without loss of generality, we suppose that $\boldsymbol{P}_i (i = 0,1,2,3) \in \mathbb{R}^2$.

3. Shape Features of Planar T-Bézier Curve

We recall the following preliminary knowledge before discussion. The interested reader is referred to [11, 18, 23, 26] for further details.

Definition 3.1 [18]. Let $\boldsymbol{r}(t)$ be a vector (or scalar) valued function such that

$$\lim_{t\to t_0+0} \boldsymbol{r}(t)/|\boldsymbol{r}(t)| = c \lim_{t\to t_0-0} \boldsymbol{r}(t)/|\boldsymbol{r}(t)|$$

where c is a constant. We say that $\boldsymbol{r}(t)$ changes direction (or sign) oppositely at t_0, if $c = -1$; and $\boldsymbol{r}(t)$ does not change direction (or sign) at t_0, if $c = 1$.

Definition 3.2 [18]. If the tangent vector $\boldsymbol{r}'(t)$ of the parametric curve $\boldsymbol{r}(t)$ changes direction oppositely at t_0, we say that the curve $\boldsymbol{r}(t)$ has a cusp at t_0.

Definition 3.3 [11, 18]. Let $\gamma(t) = \boldsymbol{r}'(t) \times \boldsymbol{r}''(t)$, where given two vectors $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$, we define the cross product $\boldsymbol{x} \times \boldsymbol{y} = x_1 y_2 - x_2 y_1$. If $\gamma(t)$ changes sign at t_0 with $\boldsymbol{r}'(t_0) \neq 0$, we say that the curve $\boldsymbol{r}(t)$ has an inflection point at t_0.

Definition 3.4 [18]. If there exists $t_1 \neq t_2$ such that $\boldsymbol{r}(t_1) = \boldsymbol{r}(t_2)$, we say that the curve $\boldsymbol{r}(t)$ has a loop.

Definition 3.5 [23]. Let

$$\gamma(t) = \boldsymbol{r}'(t) \times \boldsymbol{r}''(t),$$
$$m(t) = \boldsymbol{r}'(0) \times [\boldsymbol{r}(t) - \boldsymbol{r}(0)],$$
$$n(t) = [\boldsymbol{r}(t) - \boldsymbol{r}(0)] \times \boldsymbol{r}'(t),$$

and suppose that the curve $\boldsymbol{r}(t) (t \in [0,1])$ has no singularities. We say that the curve $\boldsymbol{r}(t)$ is globally convex if $\gamma(t)$, $m(t)$ and $n(t)$ do not change sign for all $t \in (0,1)$. We say that the curve $\boldsymbol{r}(t)$ is locally convex if $\gamma(t)$ does not change sign for all $t \in (0,1)$ while there exists $t_0 \in (0,1)$ such that $m(t)$ or $n(t)$ changes sign at t_0.

Definition 3.6 [26]. For given family of curves C_t: $F(x, y, t) = 0$ with a single parameter t, $C: f(x, y) = 0$ is called as the envelop curve of the given family of curves C_t, if the curve C satisfies that an arbitrary point $\boldsymbol{P}$ on C belongs to one curve in the given family of curves C_t and C tangents to C_t at the point $\boldsymbol{P}$.

Definition 3.7 [26]. If a set is composed of the points (x, y) that satisfy the system of equations

$$\begin{cases} F(x, y, t) = 0, \\ F_t'(x, y, t) = 0, \end{cases}$$

it is called the determining curve or discriminant of the given family of curves $F(x, y, t) = 0$.

Definition 3.8 [26]. The determining curve is called as the envelop curve of the given family of curves C_t if $\partial F/\partial x$ and $\partial F/\partial y$ are not zero at the points (x, y) on the determining curve in the same time.

Let $\boldsymbol{a}_i = \boldsymbol{P}_i - \boldsymbol{P}_{i-1} (i = 1,2,3)$, the T-Bézier curve (1) can be rewritten as

$$\boldsymbol{r}(t) = \boldsymbol{P}_0 + [1 - b_0(t)]\boldsymbol{a}_1 + [b_2(t) + b_3(t)]\boldsymbol{a}_2 + b_3(t)\boldsymbol{a}_3. \quad (3)$$

First, we suppose that the side vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$ are not parallel, i.e. $\boldsymbol{a}_1 \times \boldsymbol{a}_3 \neq 0$. Then the side vector $\boldsymbol{a}_2$ can be represented as the linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$, i.e. $\boldsymbol{a}_2 = u\boldsymbol{a}_1 + v\boldsymbol{a}_3$, where $(u, v) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3, \boldsymbol{a}_1 \times \boldsymbol{a}_2)/(\boldsymbol{a}_1 \times \boldsymbol{a}_3)$. The coefficients u and v clearly indicate the relative position of the control polygon's side vectors.

The following theorem shows the relation between the position of point (u, v) in uv-plane and the shape features of the curve (3).

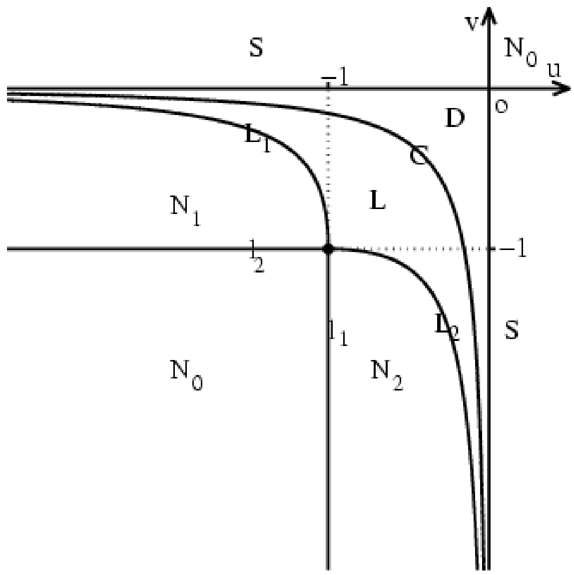

Fig. 1 Shape diagram of planar T-Bézier curve with two shape parameters.

Theorem 3.1 Assume that $\boldsymbol{a}_2 = u\boldsymbol{a}_1 + v\boldsymbol{a}_3$ with $\boldsymbol{a}_1 \times \boldsymbol{a}_3 \neq 0$. Then, the shape features of the planar T-Bézier curve $\boldsymbol{r}(t)$ are completely determined by the position of point (u, v) in uv-plane (see Fig. 1), i.e.

$$(\lambda,\mu)\in\begin{cases} N_0:\boldsymbol{r}(t) \text{ is globally convex and has no inflection points and singular points,}\\ N_1\cup N_2:\boldsymbol{r}(t) \text{ is locally convex and has no inflection points and singular points,}\\ S:\boldsymbol{r}(t) \text{ has one inflection point and no singular points,}\\ D:\boldsymbol{r}(t) \text{ has two inflection points and no singular points,}\\ C:\boldsymbol{r}(t) \text{ has one cusp and no loops and inflection points,}\\ L:\boldsymbol{r}(t) \text{ has one loop and no cusps and inflection points.}\end{cases}$$

The regions mentioned above are defined as:

$N_0=\{(u,v)\mid u\le -1, v\le -1\}\cup\{(u,v)\mid 0\le u, 0\le v\}$,

$N_1=\{(u,v)\mid k_1(u,v)<0, -1<v<0\}$,

$N_2=\{(u,v)\mid k_2(u,v)<0, -1<u<0\}$,

$S=\{(u,v)\mid uv<0\}\cup\{(0,v)\mid v<0\}\cup\{(u,0)\mid u<0\}$,

$D=\{(u,v)\mid k(u,v)>0, u<0, v<0\}$,

$C=\{(u,v)\mid k(u,v)=0, -\infty<u,v<0\}$,

$L=\{(u,v)\mid k(u,v)<0, k_1(u,v)\ge 0, k_2(u,v)\ge 0\}$,

where the associated implicit equations are:

$$k(u,v):=g^2(u,\lambda)+g^2(v,\mu)-1=0, -\infty<u,v<0, \tag{4}$$

$$k_1(u,v):=h_1^2(u,\lambda)+h_2^2(v,\mu)-1=0, -\infty<u\le -1, -1\le v<0, \tag{5}$$

$$k_2(u,v):=h_1^2(v,\mu)+h_2^2(u,\lambda)-1=0, -1\le u<0, -\infty<v\le -1, \tag{6}$$

in which

$$g(x,y):=\begin{cases}1/(1-x), & y=0,\\ (1-x+2y)/3y-[(1-x+2y)^2-3y(y+2)]^{\frac{1}{2}}/3y, & y\ne 0,\end{cases}$$

$$h_1(x,y):=\begin{cases}2/(1-x), & y=0,\\ (1-x+2y)/2y-[(1-x+2y)^2-4y(y+2)]^{\frac{1}{2}}/2y, & y\ne 0,\end{cases}$$

$$h_2(x,y):=\begin{cases}(1+x)/(1-x), & y=0,\\ (1-x+y)/2y-[(1-x+y)^2-4y(x+1)]^{\frac{1}{2}}/2y, & y\ne 0.\end{cases}$$

Proof. Substituting $\boldsymbol{a}_2=u\boldsymbol{a}_1+v\boldsymbol{a}_3$ into (3), we have

$$\begin{aligned}\boldsymbol{r}(t)=\boldsymbol{P}_0&+\{1-b_0(t)+u[b_2(t)+b_3(t)]\}\boldsymbol{a}_1\\ &+\{b_3(t)+v[b_2(t)+b_3(t)]\}\boldsymbol{a}_3.\end{aligned}\tag{7}$$

The following proof is composed of four parts corresponding to the case of cusps, inflection points, loops and convexity, respectively.

3.1. The Case of Cusps

According to Definition 3.2, the necessary condition that the curve $\boldsymbol{r}(t)$ has cups is $\boldsymbol{r}'(t)=0 (0<t<1)$. From (7), we have

$$\{-b_0'(t)+u[b_2'(t)+b_3'(t)]\}\boldsymbol{a}_1+\{b_3'(t)+v[b_2'(t)+b_3'(t)]\}\boldsymbol{a}_3=0 .$$

Since $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$ are linearly independent, letting coefficients of the vectors in $\boldsymbol{r}'(t)$ be zero, we obtain

$$\begin{cases}-b_0'(t)+u[b_2'(t)+b_3'(t)]=0,\\ b_3'(t)+v[b_2'(t)+b_3'(t)]=0.\end{cases}\tag{8}$$

From (2) and (8), we obtain parametric equations of the curve C.

$$C:\begin{cases}u(t,\lambda)=-[1-s(t)][2+\lambda-3\lambda s(t)]/2s(t),\\ v(t,\mu)=-[1-c(t)][2+\mu-3\mu c(t)]/2c(t),\end{cases} 0<t<1. \tag{9}$$

From (9), using $s^2(t)+c^2(t)=1$, we can get the implicit equation (4). The implicit form (4) is more useful when determining on which side of the curve the point (u,v) lies, while the parametric form (9) is more useful for displaying the curve.

Conversely, suppose that the point (u_0,v_0) lies on the curve C and $u_0=u(t_0,\lambda)$, $v_0=v(t_0,\mu)$, where $t_0\in(0,1)$, then $\boldsymbol{r}''(t_0)\ne 0$. Otherwise, after the discussion similar to that of (8) and (9), we can get two contradictory equations: $c^3(t_0)[2+\lambda-3\lambda s^2(t_0)]=0$ and $s^3(t_0)[2+\mu-3\mu c^2(t_0)]=0$. Therefore, according to the Taylor expansion

$$\boldsymbol{r}'(t)=\boldsymbol{r}''(t_0)(t-t_0)+o(t-t_0),$$

we know that $\boldsymbol{r}'(t)$ changes direction oppositely at t_0.

Hence we have proved the following lemma.

Lemma 3.1 The T-Bézier curve $\boldsymbol{r}(t)$ has a cusp if and only if $(u,v)\in C$, where C is determined by (5) or (9).

3.2 The Case of Inflection Points

By directly computing from (7), we can get

$$\gamma(t)=\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=f(t;u,v)\boldsymbol{a}_1\times\boldsymbol{a}_3,$$

where

$$f(t;u,v)=-\begin{vmatrix}b_0'(t) & b_3'(t)\\ b_0''(t) & b_3''(t)\end{vmatrix}+u\begin{vmatrix}b_2'(t) & b_3'(t)\\ b_2''(t) & b_3''(t)\end{vmatrix}+v\begin{vmatrix}b_0'(t) & b_1'(t)\\ b_0''(t) & b_1''(t)\end{vmatrix}.$$

According to Definitions 3.1 and 3.3, the point $\boldsymbol{r}(t_0)$ is an inflection point if and only if $f(t;u,v)$ changes sign at t_0. In the uv-plane, the possible region of inflection points must be covered by the family of straight lines $f(t;u,v)=0$. After solving the simultaneous equations $f(t;u,v)=0$ and $f_t'(t;u,v)=0$ with respect to u and v, we obtain (9).

According to Definitions 3.6, 3.7 and 3.8, the curve C is just the envelope of the family of straight lines. The curve C is strictly convex continuous curve, so that the region swept by the tangent line of the curve C is

$S \bigcup D$ (see Fig. 1), i.e. the possible region that results in inflection points.

Apparently, the curve C has at least a tangent line $f(t_0;u,v)=0$ passing through an arbitrary point (u_0,v_0) located in $S \bigcup D$. Note that $f_t'(t_0;u_0,v_0) \neq 0$ when $(u_0,v_0) \in S \bigcup D$. Otherwise, according to Definition 3.7 on the envelope, we get $(u_0,v_0) \in C$. Therefore, from the Taylor expansion $f(t;u_0,v_0) = f_t'(t_0;u_0,v_0)(t-t_0)+o(t-t_0)$, we know that $f(t;u_0,v_0)$ changes sign at $t=t_0$, consequently, the point $\boldsymbol{r}(t_0)$ is an inflection point.

Furthermore, if $(u_0,v_0) \in S$, the curve $\boldsymbol{r}(t)$ has a single inflection point because there exists a unique tangent line of the curve C passing through the point (u_0,v_0); else if $(u_0,v_0) \in D$, then the curve $\boldsymbol{r}(t)$ has two inflection points because there exist two tangent lines of the curve C passing through the point (u_0,v_0). Hence we obtain the following lemma.

Lemma 3.2 The T-Bézier curve $\boldsymbol{r}(t)$ has one (or two) inflection point if and only if $(u_0,v_0) \in S$ (or D).

3.3 The Case of Loops

From Definition 3.4, the sufficient and necessary condition that the T-Bézier curve $\boldsymbol{r}(t)$ has loops is that there exists $0 \le t_1 < t_2 \le 1$ such that $\boldsymbol{r}(t_1)-\boldsymbol{r}(t_2)=0$. According to (7), it is equivalent to u, v, t_1 and t_2 satisfy the system of equations:

$$\begin{cases} u = \dfrac{b_0(t_2)-b_0(t_1)}{b_2(t_2)+b_3(t_2)-b_2(t_1)-b_3(t_1)}, \\ v = \dfrac{b_3(t_1)-b_3(t_2)}{b_2(t_2)+b_3(t_2)-b_2(t_1)-b_3(t_1)}, \end{cases} (t_1,t_2) \in \Delta, \qquad (10)$$

where $\Delta = \{(t_1,t_2) \in \mathbb{R}^2 \mid 0 \le t_1 < t_2 \le 1\}$. The map $F:\Delta \to F(\Delta)$ defined as (10) is a topological mapping. Therefore, the image $L = F(\Delta)$ is a simply connected region in uv-plane, its boundary curves C, L_1 and L_2 correspond to the three boundary segments of Δ : $t_1 = t_2$, $t_1 = 0$ and $t_2 = 1$ respectively, this indicates that the curve C does not belong to the region L while the curves L_1 and L_2 do. The parametric equations of L_1 and L_2 are as follows:

$$L_1: \begin{cases} u = -[2+\lambda-2\lambda s(t)-s(t)-\lambda s^2(t)]/s(t), \\ v = -[1-c(t)][1-\mu c(t)]/[1+c(t)], \end{cases} 0 < t \le 1, \ (11)$$

$$L_2: \begin{cases} u = -[1-s(t)][1-\lambda s(t)]/[1+s(t)], \\ v = -[2+\lambda-2\lambda c(t)-c(t)-\lambda c^2(t)]/c(t), \end{cases} 0 \le t < 1. \ (12)$$

It can be easily checked that (5) and (6) follow from (11) and (12), respectively.

Thus, the following lemma holds:

Lemma 3.3 The T-Bézier curve $\boldsymbol{r}(t)$ has a loop if and only if $(u,v) \in L$.

Both the curves L_1 and L_2 are monotonically decreasing and strictly convex continuous curves. the curve L_1 intersects the curve L_2 at the point $(-1,-1)$. The asymptotic line of the curve L_1 is u-axis, and that of the curve L_2 is v-axis. The two asymptotic lines of the curve C are u-axis and v-axis respectively. And the curve C does not intersect L_1 and L_2.

3.4 The case of convexity

Lemmas 3.1-3.3 imply that the T-Bézier curve $\boldsymbol{r}(t)$ has none of inflection points and singular points if the point (u,v) lies in complementary region $N = \mathbb{R}^2 \setminus C \bigcup S \bigcup D \bigcup L$. As is shown in Fig. 1, the region N can be divided into N_0, N_1, N_2, where the region N_1 is bounded by the curve L_1 and the ray $l_2: v=-1, u<-1$, the region N_2 is bounded by the curve L_2 and the ray $l_1: u=-1, v<-1$. The ray $l_i (i=1,2)$ is the tangent line of the curve $L_i (i=1,2)$ at the point $(-1,-1)$.

To distinguish a local convex curve from a global one, as mentioned in Definition 3.5, we need to consider $\gamma(t)$, $m(t)$ and $n(t)$. By a straightforward computation from (7), we have

$$m(t) = \pi(1+\lambda/2)\varphi(t;u,v)\boldsymbol{a}_1 \times \boldsymbol{a}_3$$

and

$$n(t) = \psi(t;u,v)\boldsymbol{a}_1 \times \boldsymbol{a}_3,$$

where

$$\varphi(t;u,v) = b_3(t) + v[b_2(t)+b_3(t)], \qquad (13)$$

$$\begin{aligned}\psi(t;u,v) = {} & [1-b_0(t)]b_3'(t) + b_0'(t)b_3(t) + u[b_2(t)b_3'(t) - b_2'(t)b_3(t)] \\ & + v\{[1-b_0(t)][b_2'(t)+b_3'(t)] + b_0'(t)[b_2(t)+b_3(t)]\}.\end{aligned}$$

According to (13), if $v_0 = -b_3(t_0)/[b_2(t_0)+b_3(t_0)]$, then $\varphi(t;u,v_0)$ changes sign at t_0. It can be easily checked that the range of v_0 is $-1 < v_0 < 0$. Therefore $m(t)$ changes sign at t_0, if $(u_0,v_0) \in N_1$. In fact, the region N_1 is just the part of the region N, which is covered by the tangent lines of the curve L_2 (see Fig. 1).

Parametric Equations (11) can be obtained by solving the simultaneous equations $\psi(t;u,v)=0$ and $\psi_t'(t;u,v)=0$ for the unknown parameters u and v. This implies that the region N_2 is covered by the tangent lines of the curve L_1. If the point (u_0,v_0) lies in the region N_2, then the curve L_1 has a tangent line $\psi(t_0;u,v)=0$ passing through it with $\psi_t'(t_0;u_0,v_0)\neq 0$. Thus, according to the Taylor expansion $\psi(t;u_0,v_0)=\psi_t'(t_0;u_0,v_0)(t-t_0)+o(t-t_0)$, we know that $\psi(t;u_0,v_0)$ changes sign at t_0.

In summary, $\gamma(t)$, $m(t)$ and $n(t)$ do not change sign for all $t\in(0,1)$ when $(u_0,v_0)\in N_0\cup N_1\cup N_2$, while there exits $t_0\in(0,1)$ such that $m(t)$ (or $n(t)$) changes sign at t_0 when $(u_0,v_0)\in N_1$ (or N_2). This establishes the following lemma.

Lemma 3.4 The T-Bézier curve $\boldsymbol{r}(t)$ is globally (or locally) convex if and only if $(u,v)\in N_0$ (or $N_1\cup N_2$).

Lemmas 3.1-3.4 give the desired Theorem 3.1 on the shape features of the planar T-Bézier curve $\boldsymbol{r}(t)$.

The proof is finished. □

Finally, if the side vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$ are parallel (excluding the four control points are collinear), after discussion analogous to the proof of Theorem 3.1, we can deduce that the T-Bézier curve is globally convex and has no inflection points and singularities if the direction of $\boldsymbol{a}_1$ is opposite to that of $\boldsymbol{a}_3$, the T-Bézier curve has one and only one infection points if and only if the direction of $\boldsymbol{a}_1$ is same as that of $\boldsymbol{a}_3$.

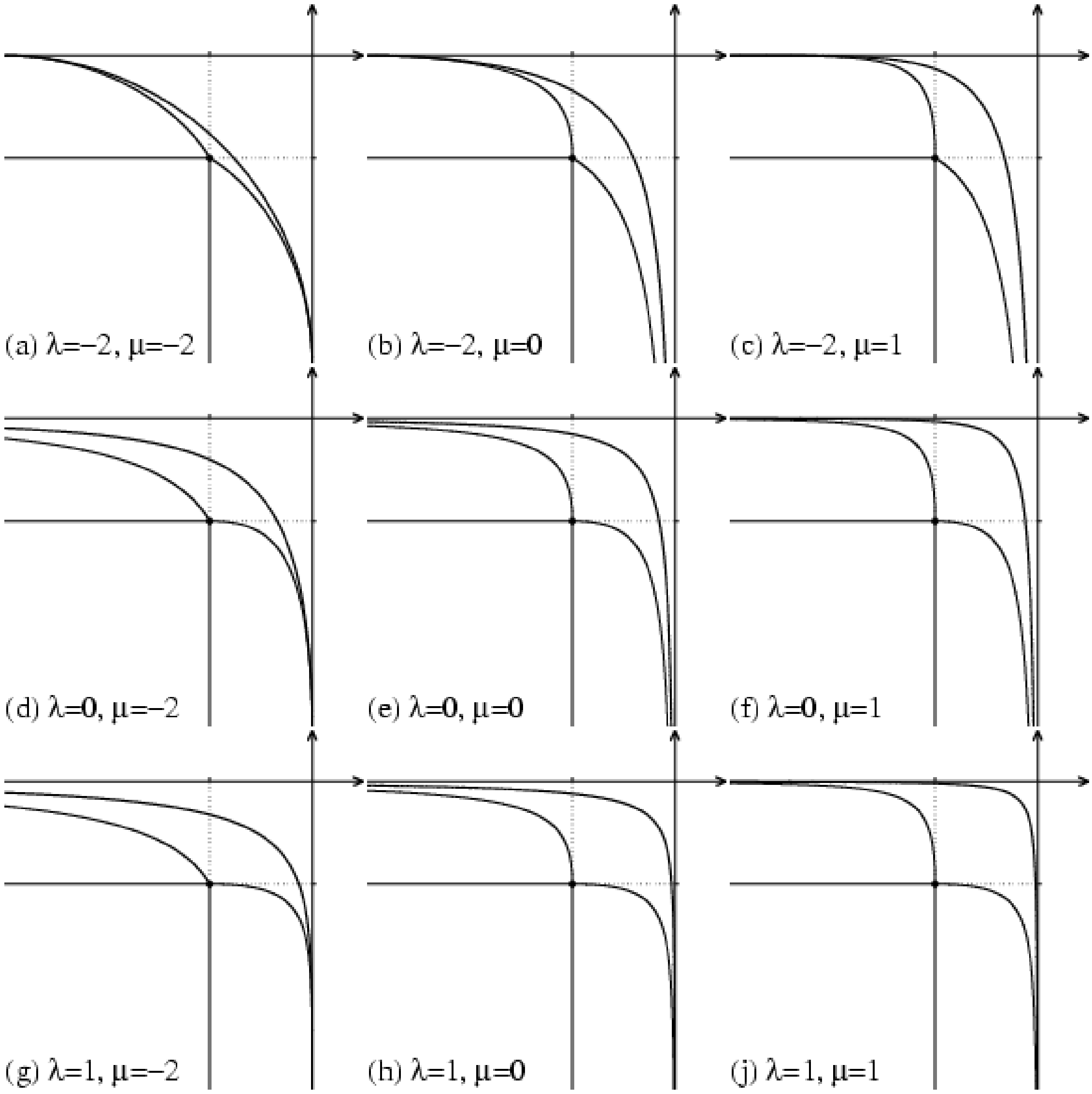

Fig. 2 The influences on shape diagram of T-Bézier curve by the two shape parameters.

4. Effects of the Shape Parameters on the Shape Features

Theorem 3.1 shows that the regions S and N_0 are independent of the shape parameters λ and μ, and the equations of the curves C, L_1 and L_2 only depend on the values of the two shape parameters. To know the changes of the regions D and L, we let $Area(X)$ denotes the area of region X in uv-plane. From the equations (9), and (11-12), we have

$$Area(D) = 1 + (2-3\pi)(\lambda+\mu) + (4-21\pi/16)\lambda\mu, \qquad (14)$$

$$Area(L) = 4/3 + (4-5\pi/4)(\lambda+\mu) + (20/3-35\pi/16)\lambda\mu. \qquad (15)$$

According to the equations of the curves C, L_1 and L_2, and the areas (14-15), we know the following changes of the above-mentioned curves and regions in terms of the shape parameters λ and μ:

Remark 4.1 Assume that λ is fixed. The curve L_1 and the left part of the curve C gradually approach u-axis as the increment of μ; but the curve L_2 and the right part of the curve C gradually approach v-axis as the decrement of μ.

Remark 4.2 Assume that μ is fixed. The curve L_1 and the left part of the curve C gradually approach u-axis as the decrement of λ. The curve L_2 and the right part of the curve C gradually approach v-axis as the increment of λ.

Remark 4.3 Assume that λ is fixed. As the increment of μ the region N_1 expands, but the regions N_2 and D reduce.

Remark 4.4 Assume that μ is fixed. As the increment of λ the region N_2 expands, but the regions N_1 and D reduce.

Remark 4.5 Assume that λ is fixed. The region L expands as the increment of μ, if $\lambda < \lambda^*$. The region L reduces as the increment of μ, if $\lambda > \lambda^*$.

Remark 4.6 Assume that μ is fixed. The region L expands as the increment of λ, if $\mu < \mu^*$. The region L reduces as the increment of λ, if $\mu > \mu^*$.

Note that $\lambda^* = \mu^* = (192/5 - 12\pi)/(21\pi - 64) \approx 0.3552$ can be derived form (15).
These changes are shown in Fig. 2.

Corollary 4.1 When there is only a single inflection point on the T-Bézier curve $\boldsymbol{r}(t)$, we cannot remove it by altering the shape parameters. And if $\boldsymbol{r}(t)$ is globally convex, it remains global convexity regardless of the changes of shape parameters λ and μ.

Corollary 4.2 If $-1 < u, v < 0$, then the T-Bézier curve $\boldsymbol{r}(t)$ has either a singularity or two inflection points regardless of the changes of shape parameters λ and μ (see Fig. 3).

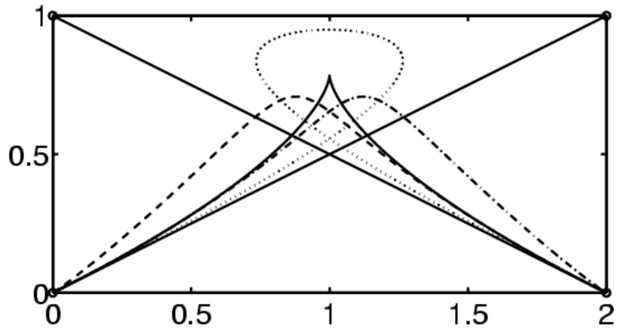

Fig. 3 The T-Bézier curve has either a singularity or two inflection points.

Corollary 4.3 Assume that $(u,v) \in N_1$ with $\lambda = -2$ and $\mu = 1$; or $(u,v) \in N_2$ with $\lambda = 1$ and $\mu = -2$. Then we can remove the unwanted singularity or inflection points of the T-Bézier curve $\boldsymbol{r}(t)$ by modifying shape parameters λ and μ (see Fig. 4).

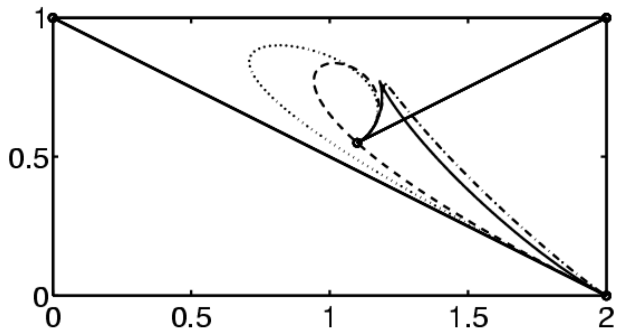

Fig. 4 The T-Bézier curve can be adjusted as local convex curve.

5. Conclusions

We investigated the convexity and existence of singularities and inflection points of planar trigonometric Bézier curves with two shape parameters. The obtained conditions enables us to manipulate the control points and the shape parameters such that the resulting curves will not have the unwanted inflection points and singularities. The effects of the shape parameter on the shape diagram of planar T-Bézier curves were made clear. The results are useful for characterizing and adjusting the shapes of planar trigonometric curves.

References

[1] X. L. Han, "Quadratic Trigonometric Polynomial Curves with A Shape Parameter", Computer Aided Geometric Design, Vol.19, No. 7, 2002, pp. 503-512.

[2] X. L. Han, "Cubic Trigonometric Polynomial Curves with A Shape Parameter", Computer Aided Geometric Design, Vol. 21 , No. 6, 2004, pp. 535-548.

[3] X. L. Han, "C^2 Quadratic Trigonometric Polynomial Curves with Local Bias", Journal of Computational and Applied Mathematics, Vol. 180, No. 1, 2005, pp. 161-172.

[4] X.A. Han, Y.C. Ma, and X .L. Huang, "The Cubic Trigonometric Bézier Curve with Two Shape Parameters", Applied Mathematics Letters, Vol. 22, No. 2, 2009, pp. 226-231.

[5] N. Choubeya, and A. Ojha, "Constrained Curve Drawing Using Trigonometric Splines Having Shape Parameters", Computer-Aided Design, Vol. 39, No. 12, 2007, pp. 1058-1064.

[6] L. L. Yan, and J. F. Liang, "A Class of Algebraic-Trigonometric Blended Splines", Journal of Computational and Applied Mathematics, Vol. 235, No. 6, 2011, pp. 1713-1729.

[7] P. E. Koch, T. Lyche , M. Neamtu and L. L. Schumaker, "Control Curves and Knot Insertion for Trigonometric Splines", Advance in Computational Mathematics, Vol. 3, No. 4, 1995, pp. 405-424.

[8] J. M. peña, "Shape Preserving Representations for Trigonometric Polynomial Curves", Computer Aided Geometric Design, Vol. 14, No. 1, 1997, pp. 5-11.

[9] P. Alfeld, M. Neamtu and L. Schumaker, "Fitting Scattered Data On Sphere Like Surfaces Using Spherical Splines", Journal of Computational and Applied Mathematics, Vol. 73, No. 1, 1996, pp. 5-43.

[10] G. Walz, "Trigonometric Bézier and Stancu Polynomials over Intervals and Triangles", Computer Aided Geometric Design, Vol. 14, No. 4, 1997, pp. 393-397.

[11] D. Manocha, and J. F. Canny, "Detecting Cusps and Inflection Points in Curves", Computer Aided Geometric Design, Vol. 9, No. 1, 1992, pp. l-24.

[12] I. Juhász, "On the Singularity of A Class of Parametric Curves", Computer Aided Geometric Design, Vol. 23, No. 2, 2006, pp. 146-156.

[13] C. Wang, "Shape Classification of the Parametric Cubic Curve and Parametric B-spline Cubic Curve", Computer Aided Design, Vol. 13, No. 4, 1981, pp. 199-206.

[14] B. Su, and D. Liu, "An Affine Invariant and Its Application in Computational Geometry", Scientia Sinica (Series A), Vol. 24, No. 3, 1983, pp. 259-267.

[15] M. Stone, and T. DeRose, "A Geometric Characterization of Parametric Cubic Curves", ACM Transactions on Graphics, Vol. 8, No. 3, 1989, pp. 147-163.

[16] D. S. Meek and D. J. Walton, "Shape Determination of Planar Uniform Cubic B-spline Segments", Computer-Aided Design, Vol. 22, No. 7, 1990, pp. 434-441.

[17] D. S. Kim, "Hodograph Approach to Geometric Characterization of Parametric Cubic Curves", Computer Aided Design, Vol. 25, No. 10, 1993, pp. 644-654.

[18] Z. L. Ye, "Shape Classification of Planar B-spline Cubic Curves", ADVANCES IN CAD/CAM, Proceedings of the Sino-German CAD/CAM Conference, Xi'an, P.R. of China, 1987, pp. 188-196.

[19] Y. Li and R. J. Cripps, "Identification of Inflection Points and Cusps on Rational Curves", Computer Aided Geometric Design, Vol. 14, No. 5, 1997, pp. 491-497.

[20] M. Sakai, "Inflection Points and Singularities on Planar Rational Cubic Curve Segments", Computer Aided Geometric Design, Vol. 16, No. 3, 1999, pp. 149-156.

[21] J. Monterde, "Singularities of Rational Bézier Curves", Computer Aided Geometric Design, Vol. 18, No. 8, 2001, pp. 805-816.

[22] Q. Yang and G. Wang, "Inflection Points and Singularities on C-curves", Computer Aided Geometric Design, Vol. 21, No. 4, 2004, pp. 207-213.

[23] C. Liu and C.R. Traas, "On Convexity of Planar Curves and Its Application in CAGD", Computer Aided Geometric Design, Vol. 14, No. 7, 1997, pp. 653-669.

[24] R. J. Wu, "Shape Analysis of Planar Cubic H-Bézier Curve", Acta Mathematicae Applicatae Sinica (in Chinese), vol. 30, No. 5, 2007, pp. 816-821.

[25] X. A. Han, X. L. Huang and Y. C. Ma, "Shape Analysis of Cubic Trigonometric Bézier Curves with A Shape Parameter", Applied Mathematics and Computation, Vol. 217, No. 10, 2010, pp. 2527-2533.

[26] J. W. Bruce and P. J. Giblin, "Curves and Singularities: A Geometrical Introduction to Singularity Theory", New York , Cambridge University Press, 1993, pp. 73-83.