A Generalized symmetric single step method for solving interval linear systems.

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Abstract

Systems of algebraic equations with interval coefficients are very common in several areas of engineering sciences. Generalized intervals extend classical intervals providing better algebraic properties. These properties allow one constructing a generalized symmetric single step method. This paper proposes a new C-XSC (C- for eXtended Scientific Computing) software for the symmetric single step method with generalized intervals for computing an enclosure for the solution set. Examples illustrating the applicability of the proposed method are solved, and compared with other methods.

Keywords: *interval linear systems, validated interval software, C-XSC, symmetric single step method.*

1. Introduction

Solving linear systems involving uncertainties in the parameters is an important part of the solution to many scientific and engineering problems. But in real life situations, parameters of these systems often are charged by different kinds of uncertainties[4, 5, 6]. Leontief's input output model of economy[16] can be taken as an example[29]. In many cases, uncertainty can be represented by intervals. Since the seminal publication by Moore's[17], a rapid development of interval arithmetic had been observed. The system of linear interval equations can be presented as follows:

$$[A]\cdot x=[b], \qquad (1)$$

with $x \in \mathbb{R}^n$, the matrix $[A] \in \mathbb{IR}^{n\times n}$ and the vector $[b] \in \mathbb{IR}^n$ are said to belong to interval family if their elements are from some real intervals. System (1) is called the interval system of equations. Suppose $[A]$ is regular i.e. A is nonsingular for any $A \in [A]$. Then for a matrix $A \in [A]$ and any vector $b \in [b]$ an ordinary linear system $A \cdot x = b$ has the unique solution. We are interested in a set $\sum$ of all these solutions of interval system:

$$\sum = \{x \in \mathbb{R}^n : Ax = b, A \in [A], \quad b \in [b]\}$$

The characterization of this set has been obtained in[20]. It has been proved that the intersection of $\sum$ with each orthant in $\Re^n$ gives a convex polytope. But in general, $\sum$ is non-convex as the union of convex sets, and its detailed description meets combinatorial difficulties. The main objective is to find interval solution of linear interval system that is to determine the smallest interval vector [y] containing all possible solutions. In other words, we need to imbed the solution set $\sum$ into the minimal box in $\mathbb{R}^n$. This problem is known to be NP-hard [15] and complicated from computational viewpoint for large-scale systems. Oettli[21] shows how multiple linear programming can be used to obtain [y]; this line of research was continued by Cope and Rust[7], and Rust and Burrus[26]. Some iterative approaches were established t this context as well as direct numerical methods that provide over-bounding of [y] (see monographs[8, 9, 19] and papers[25, 27]).

In this paper, we propose a new C-XSC software (C for Extended Scientific Computing)[10] of the symmetric single step method for the solution of the systems of interval linear equations with the use of „generalized interval". We will compare our method to other methods. The rest of the paper is set as follows. In Section 2, some Basic notation is introduced. In section 3, we recall the fundamentals of „generalized interval" and present its interpretation as the modified of interval arithmetic. The main results of this paper is presented in Section 4. Another modification for the symmetric single step method is introduced in Section 5. Numerical and practical examples illustrating the features of the proposed method are provided in Section 6. Last section concludes with some remarks.

2. Basic notations

We use the following notations $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times n}$, $\mathbb{IR}, \mathbb{IR}^n, \mathbb{IR}^{n\times n}$, to denote the set of real numbers, the set of real vectors with *n* components, the set of real *n*×*n* matrices, the set of intervals, the set of interval vectors with *n* components and the set of *n*×*n* interval matrices, respectively. By interval we mean a real compact interval

$$[x] := [a, b] := \{x \in \mathbb{R} \mid \quad a \le x \le b\}$$

For $[x], [y] := [c, d]$ we define

- The mid-point $\mathrm{mid}([x]) := {}^{(a+b)}/_{2}$,
- the radius $\mathrm{rad}([x]) := {}^{(b-a)}/_{2}$,
- the absolute value $|[x]| := \max\{|a|, |b|\}$, ,
- the distance $q([x], [y]) := \max\{|a - c|, |b - d|\}$

- minimal absolute value (magnitude)

$$\prec[x]\succ := \min\{|x| : x \in [x]\} = \begin{cases} \min\{|a|,|b|\} & \text{if } 0 \notin [x] \\ 0 & \text{else} \end{cases} \quad (2)$$

For interval vectors and interval matrices, these quantities are defined componentwise. If for two interval vectors $[u], [v] \in \mathbb{IR}^n$ we have $[u_i] \cap [v_i] \neq \emptyset, i = 1,2,\cdots,n,$ then $[u] \cap [v] := ([u_i] \cap [v_i])$ otherwise $[u] \cap [v] = \emptyset$. In addition, for $[u], [v] \in \mathbb{IR}^n$ we define $[u] \subseteq [v$ iff $[u_i] \subseteq [v_i], i = 1,2 \cdots, n$. Furthermore, we repeat some relations concerning the distance:

$$q([u],[v]) \leq q([u],[w]) + q([w],[v])$$
$$q([u]+[w],[v]+[w]) = q([u],[v])$$
$$q([u]+[v],[w]+[z]) \leq q([u],[w]) + q([v],[z])$$

if $[u],[u],[w],[z] \in \mathbb{IR}^n$.

For square interval matrices we define the comparison matrix (Ostrowsky matrix) $\langle [A] \rangle := (C_{ij}) \in \mathbb{R}^{n \times n}$ using (2) by setting

$$c_{ij} := \begin{cases} -|a_{ij}| & \text{if} \quad i \neq j \\ \langle [a_{ij}] \rangle & \text{if} \quad i = j \end{cases}$$

A square matrix $[A] \in \mathbb{IR}^{n \times n}$ is called regular if all $A \in [A]$ are nonsingular.

If $\text{mid}([A]) \cdot [A]$ is regular then $[A]$ is strongly regular. An interval matrix $[A]$ is an H-matrix iff there exist a vector $v > 0$ such that $\langle [A] \rangle v > 0$.

Definition 1 [28] *Let $A, B, C \in \mathbb{R}^{n \times n}$. Then $A = B - C$ is a regular splitting of A if $C \geq 0$ and B is nonsingular with $B^{-1} \geq 0$.*

Theorem 1 [28] *Assume that is $A \in \mathbb{R}^{n \times n}$ nonsingular, that $A^{-1} \geq 0$ and that $A = B - C$ is a regular splitting of A. Then $\rho(B^{-1}C) < 1$, where $\rho(\ )$ denotes the spectral radius of a matrix.*

Regular splitting was introduced in [28], where one can also find the proof of Theorem 1.

3. Generalized intervals

Generalized intervals are intervals whose bounds are not constrained to be ordered, for example [-2,2] and [2,-2] are generalized intervals. They have been introduced in[12,22] so as to improve the algebraic structure of intervals, while maintaining the inclusion monotonicity. The set of generalized intervals is denoted by $\mathbb{KR}$ and is divided into three subset:

- The set of *proper intervals* with bounds ordered increasingly. These proper intervals are identified with classical intervals. The set of proper intervals is denoted $\mathbb{IR} := \{[a,b] | a \leq b\}$. *Strictly* proper intervals satisfy $a < b$.
- The set of *improper intervals* with bounds ordered decreasingly. It is denoted by $\overline{\mathbb{IR}} := \{[a,b] | a \geq b\}$. *Strictly* improper intervals satisfy $a > b$.
- The set of *degenerated intervals* $\{[a,b] | a = b\} = \mathbb{IR} \cap \overline{\mathbb{IR}}$. Degenerated intervals are identified to reals.

Therefore, form a set of reals $\{x \in \mathbb{R}^n | a \leq x \leq b\}$, one can build the two generalized intervals [a,b] and [b,a]. It will be convenient to switch from one to the other keeping the underlying set of reals unchanged. To this purpose, the following three operations are introduced:

- The dual operation is defined by $\text{dual}([a,b]) = [b,a]$.
- The proper projection is defined by $\text{pro}([a,b]) = [\min\{a,b\}, \max\{a,b\}]$.
- The improper projection is defined by $\text{imp}([a,b]) = [\max\{a,b\}, \min\{a,b\}]$.

The generalized intervals are partially ordered by an inclusion which extends the inclusion of classical intervals. Given two generalized intervals $[x] = [\underline{x}, \overline{x}]$ and $[y] = [\underline{y}, \overline{y}]$, the inclusion is defined by $[x] \subseteq [y] \Leftrightarrow \underline{y} \leq \underline{x} \wedge \overline{x} \leq \overline{y}$. For example, $[-1,1] \subseteq [-1.1,1.1]$ (this matches the set inclusion), $[-1.1,1.1] \subseteq [-1,1]$ (the inclusion between the underlying set of real is reversed for improper intervals) and $[2,0.9] \subseteq [-1,1]$. As degenerated intervals are identified to reals, if $[x]$ proper then $x \in [x] \Leftrightarrow x \subseteq [x]$. On the other hand, if $[x]$ is strictly improper then for all $x \in \mathbb{R}$ the inclusion $x \subseteq [x]$ is false.

The generalized interval arithmetic (Kaucher arithmetic) extends the classical interval arithmetic. Its definition can be found in[13,27]. When only proper intervals are involved, this arithmetic coincides with the interval arithmetic: $[x] \circ [y] = \{x \circ y \in \mathbb{R} | x \in [x], y \in [y]\}$. When proper and improper intervals are involved, some new expressions are used . For example, $[a,b] + [c,d] = [a+c, b+d]$ and if $a,b,c,d \geq 0$ then $[a,b] \cdot [c,d] = [a \cdot c, b \cdot d]$. The following useful property provides some bounds on the proper projection of the results of the generalized interval arithmetic. Let us consider $[x],[y] \in \mathbb{KR}$ and $\circ \in \{+,-,\cdot,/\}$. If $\text{pro}[x] \circ \text{pro}[y]$ is defined then $[x] \circ [y]$ is defined and it satisfies

$$\text{pro}([x] \circ [y]) \subseteq (\text{pro}[x]) \circ (\text{pro}[y])$$

Generalized interval arithmetic has better algebraic properties than the classical interval arithmetic. The addition in $\mathbb{KR}$ is a group. The opposite of an interval $[x]$ is – dual$[x]$, i.e.

$$[x] + (-\text{dual}[x]) = [x] - \text{dual}[x] = [0,0].$$

The multiplication in $\mathbb{KR}$ restricted to generalized intervals whose proper projection does not contain 0 is also a group. The inverse of such a generalized interval $[x]$ is $1/\text{dual}[x]$, i.e.,

$$[x] \cdot (1/\text{dual}[x]) = [x]/(\text{dual}[x]) = [1,1] .$$

Although addition and multiplication in $\mathbb{KR}$ are associative, they are not distributive. The addition and multiplication in $\mathbb{KR}$ are linked by the following distributivity laws[24,27]. Whatever are $[x],[y],[z] \in \mathbb{KR}$

- Conditional distributivity:
  $[x]\cdot[y]+(\mathrm{imp}[x])\cdot[z] \subseteq [x]\cdot([y]+[z]) \subseteq [x]\cdot[y]+(\mathrm{pro}[x])\cdot[z]$.

The three following particular cases will be of practical interest in this paper.

- Subdistributivity: if $[x] \in \mathbb{KR}$ then $[x]\cdot([y]+[z]) \subseteq [x]\cdot[y]+[x]\cdot[z]$;
- Superdistributivity: if $[x] \in \overline{\mathbb{KR}}$ then $[x]\cdot([y]+[z]) \supseteq [x]\cdot[y]+[x]\cdot[z]$;
- Distributivity: if $x \in \mathbb{R}$ then $x\cdot([y]+[z]) = x\cdot[y]+x\cdot[z]$.

Another useful property of the Kaucher arithmetic is its monotonicity with respect to the inclusion. Whatever are $\circ \in \{+,-,\cdot,/\}$ and $[x],[y],[xx],[yy] \in \mathbb{KR}$,
$[x] \subseteq [xx] \cap [y] \subseteq [yy] \Longrightarrow ([x]\circ[y]) \subseteq ([xx]\circ[yy])$.
The next example illustrates the way these properties will be used in the sequel.

Example 1: *Consider the expression* $[x]+[u][v] \subseteq [y]$. *Subtracting* $\mathrm{dual}([u][v]) = (\mathrm{dual}[u])(\mathrm{dual}[v])$ *to each side preserves the inclusion:* $[x]+[u][v]-\mathrm{dual}([u][v]) \subseteq [y]-(\mathrm{dual}[u])(\mathrm{dual}[v])$. *As* $-\mathrm{dual}([u][v])$ *is the opposite of* $[u][v]$, *the following inclusion is eventually proved to hold:* $[x] \subseteq [y]-(\mathrm{dual}[u])(\mathrm{dual}[v])$.

Finally, generalized interval vectors $[x] \in \mathbb{KR}^n$ and generalized interval matrices $[A] \in \mathbb{KR}^{n\times n}$ together with their additions and multiplications are defined similarly to their real and classical interval counterparts.

4. Generalized symmetric single step method

In this section we assume that the reader is familiar with the concept P contractions for proving the convergence of a fixed point iteration to a unique fixed point for an arbitrary starting vector. For the details please see [1,2].

We assume throughout that the matrix $[A]$ is nonsingular, and moreover that its diagonal entries $[a_{ii}]$ are all nonzero. We can express the matrix $[A]$ as the matrix sum [1,28]

$$[A] = [D]+[L]+[U],$$

Where $[D]$ is a diagonal matrix, and $[L]$ and $[U]$ are respectively strictly lower and upper triangular matrices. We can write (1) as

$$[D]\cdot x = [b]-([L]+[U])\cdot x$$

Then

$$x = [D]^{-1}\cdot([b]-([L]+[U]\cdot x), \quad (3)$$

where

$$[L] = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ [a_{21}] & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ [a_{n1}] & \cdots & [a_{nn-1}] & 0 \end{pmatrix} \quad (4)$$

$$[U] = \begin{pmatrix} 0 & [a_{12}] & \cdots & [a_{1n}] \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & [a_{n-1n}] \\ 0 & 0 & \ldots & 0 \end{pmatrix} \quad (5)$$

$$[D]^{-1} = \begin{pmatrix} \frac{1}{\mathrm{dual}[a_{11}]} & 0 & \cdots & 0 \\ 0 & \frac{1}{\mathrm{dual}[a_{22}]} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{\mathrm{dual}[a_{nn}]} \end{pmatrix} \quad (6)$$

Since the diagonal entries $[a_{ii}]$ of $[A]$ are nonzero, we can carry out the following iterative method derived from (3):

$$\left[x_i^{(l+1/2)}\right] = \frac{\left([b_i]-\sum_{j=1}^{i-1}\mathrm{dual}\left([a_{ij}]\left[x_i^{(l+1/2)}\right]\right)-\sum_{j=i+1}^{n}\mathrm{dual}\left([a_{ij}]\left[x_i^{(l)}\right]\right)\right)}{\mathrm{dual}([a_{ii}])}, \quad 1 \le i \le n \quad (7)$$

$$\left[x_i^{(l+1)}\right] = \frac{\left([b_i]-\sum_{j=1}^{i-1}\mathrm{dual}\left([a_{ij}]\left[x_i^{(l+1/2)}\right]\right)-\sum_{j=i+1}^{n}\mathrm{dual}\left([a_{ij}]\left[x_i^{(l+1)}\right]\right)\right)}{\mathrm{dual}([a_{ii}])}, \quad 1 \le i \le n,\ l \ge 0 \quad (8)$$

where the $[x^{(0)}]'s$ initial interval vector. We call this iteration procedure the **generalized symmetric single step method**.

Theorem 2. *Consider interval linear system (1), we define* $[L]$, $[U]$ *and* $[D]^{-1}$ *as in (4), (5) and (6), respectively. Then, the sequence* $\left[x^{(l)}\right]_{l=0}^{\infty}$ *calculated according to the iteration method (generalized symmetric single step method defined as in (7) and (8)), converges for all interval vectors* $[x^{(0)}] \in \mathbb{IR}^n$ *to* $[x^*]$, *where* $[x^*]$ *is the unique fixed point of the equation (3).*

5. Generalized symmetric single step method with intersection

In this section we consider modifications of the preceding iterative methods which are based on the fact that if for any of these methods one is starting with an interval vector containing the limit, then all iterates contain the limit. Therefore the enclosure of the limit might be improved by forming intersections after each iteration step.

Theorem 3. *Let* $[A] \in \mathbb{IR}^{n\times n}$ and $[b] \in \mathbb{IR}^n$ *be given.* $[L]$, $[U]$ *and* $[D]^{-1}$ *as in (4), (5) and (6), respectively. Let* $[x^*]$ *is the unique fixed point of the equation (3). We assume that we have an interval vector* $[\text{initial}] \in \mathbb{IR}^n$ *satisfying* $[x^*] \subseteq [\text{initial}]$. *We consider the generalized symmetric single step method with intersection.*

$$\begin{cases} [x^{(0)}] := [\text{initial}] \\ \textbf{for } i = 1 \textbf{ to } n \textbf{ do} \\ \left[x_i^{(l+1/2)}\right] := [x_i^{(l)}] \cap \left\{ \left([b_i] - \sum_{j=1}^{i-1} \text{dual}\left([a_{ij}]\left[x_i^{(l+1/2)}\right]\right) - \sum_{j=i+1}^{n} \text{dual}([a_{ij}][x_i^{(l)}]) \right) \Big/ \text{dual}([a_{ii}]) \right\} \\ \textbf{for } i = 1 \textbf{ to } n \textbf{ do} \\ [x_i^{(l+1)}] := \left[x_i^{(l+1/2)}\right] \cap \left\{ \left([b_i] - \sum_{j=1}^{i-1} \text{dual}\left([a_{ij}]\left[x_i^{(l+1/2)}\right]\right) - \sum_{j=i+1}^{n} \text{dual}([a_{ij}][x_i^{(l+1)}]) \right) \Big/ \text{dual}([a_{ii}]) \right\} \end{cases} \quad (9)$$

Then $\lim_{l\to\infty}[x^{(l)}] = [x^*]$.

To get an interval vector [initial]. We assume that $[A] \in \mathbb{IR}^{n\times n}$ is an H-matrix. Let $[L]$, $[U]$ *and* $[D]^{-1}$ and $[x^*]$ defined as in Theorem 2. Then we consider symmetric single step method with arbitrary $[x^{(0)}]$. We assume that $P := \langle [D] \rangle^{-1}|[L] + [U]|$ where $\rho(P) < 1$ (see Theorem 11.4 in [1]). For $m > l$ we get [2]:

$$\begin{aligned} q\left(\left[x^{(m)}\right],\left[x^{(l)}\right]\right) &\le q\left(\left[x^{(m)}\right],\left[x^{(m-1)}\right]\right) + \cdots + q\left(\left[x^{(l+1)}\right],\left[x^{(l)}\right]\right) \\ &\le P^{m-1} \cdot q\left(\left[x^{(m)}\right],\left[x^{(0)}\right]\right) + \cdots + P^{l} \cdot q\left(\left[x^{(1)}\right],\left[x^{(0)}\right]\right) \\ &= P^{l} \cdot \left(I + P + \cdots + P^{m-l-1}\right) \cdot P^{m-1} \cdot q\left(\left[x^{(1)}\right],\left[x^{(0)}\right]\right) \\ &\le P^{l} \cdot \left(\sum_{i=0}^{\infty} P^{i}\right) \cdot P^{m-1} \cdot q\left(\left[x^{(1)}\right],\left[x^{(0)}\right]\right) \\ &= P^{l} \cdot (I-P)^{-1} \cdot P^{m-1} \cdot q\left(\left[x^{(1)}\right],\left[x^{(0)}\right]\right). \end{aligned}$$

Since $\lim_{l\to\infty}[x^{(l)}] = [x^*]$, it holds that (set $m := 1$)

$$q\left(\left[x^*\right],\left[x^{(l)}\right]\right) \le P \cdot (I-P)^{-1} \cdot q\left(\left[x^{(1)}\right],\left[x^{(0)}\right]\right) =: u,$$

Then

$$\underline{x}^{(l)} - u \le \underline{x}^*, \quad \overline{x}^* \le \overline{x}^{(l)} + u$$

Hence, we get $[x^*] \subseteq \left[\underline{x}^{(l)} - u, \overline{x}^{(l)} + u\right] := [\text{initial}]$

Algorithm 1. ***Interval linear systems (H-Matrix)***

1. Computation of an initial interval vector

$$P := \langle [D] \rangle^{-1}|[L] + [U]|,$$

$$[\text{initial}] := \left[\underline{x}^{(l)} - u, \overline{x}^{(l)} + u\right], \quad l > 0$$

2. Verification step

$\left[x^{(1)}\right] = [\text{initial}]$

repeat

if intersection = 0 **then**

Using equations (7) and (8)

else Using equation (9)

until $[x^{(l+1)}]$ *and* $[x^{(l)}]$ *are equals*

3.

if $[x^{(l+1)}]$ *and* $[x^{(l)}]$ *are equals* **then**

$\hat{x} \in [x^{(l+1)}]$ ($\hat{x}$ the exact solution)

else *no inclusion can be computed*

6. numerical examples

Example 1:

Consider

$$\begin{pmatrix} [0.4481568, 0.4498432] & [0.4376422, 0.4393578] \\ [0.4376938, 0.4393062] & [0.6503902, 0.6516098] \end{pmatrix} \cdot x = \begin{pmatrix} [0.5646710, 0.5667290] \\ [0.6103170, 0.6134830] \end{pmatrix}$$

Proposed method	Formal-Algebraic [3]	Hoelbig [11]
[0.9745537,1.0242698]	[0.974,1.0246]	[0.9740262,1.02468993]
[0.2460811,0.2874072]	[0.2458,0.2877]	[0.24574949,0.28783412]

Example 2:

Consider

$$\begin{pmatrix} 3 & [0,1] & [0,1] \\ [0,1] & 3 & [0,1] \\ [0,1] & [0,1] & 3 \end{pmatrix} \cdot x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Proposed method	Hoelbig [11]
[0.111111,0.333334]	[0.04394499,0.45605500]
[0.111111,0.333334]	[0.04394499,0.45605500]
[0.111111,0.333334]	[0.04394499,0.45605500]

Example 3:

Consider

$$\begin{pmatrix} [1.6,1.8] & [0.4,0.5] & [0.2,0.3] \\ 0 & [0.6,0.8] & [0.2,0.3] \\ 0.1 & 0 & [1.4,1.8] \end{pmatrix} \cdot x = \begin{pmatrix} [1.4,2] \\ [3,3.4] \\ [2,2.3] \end{pmatrix}$$

Proposed method	Hoelbig [11]
[-1.10487815,0.33737055]	[-1.08943500,0.52626809]
[3.10433362,5.30254389]	[2.75007232,5.42039269]
[1.09236830,1.72177701]	[0.99326595,1.72943197]

Example 4: Application

we consider a linear resistive network, presented in [30].

$$\begin{pmatrix} [1.98,2.02] & [-1.01,-0.99] & 0 & 0 & 0 \\ [-1.01,-0.99] & [2.97,3.03] & [-1.01,-0.99] & 0 & 0 \\ 0 & [-1.01,-0.99] & [2.97,3.03] & [-1.01,-0.99] & 0 \\ 0 & 0 & [-1.01,-0.99] & [2.97,3.03] & [-1.01,-0.99] \\ 0 & 0 & 0 & [-1.01,-0.99] & [1.98,2.02] \end{pmatrix} \cdot v = \begin{pmatrix} 10 \\ 0 \\ 10 \\ 0 \\ 0 \end{pmatrix}$$

Proposed method	Hoelbig [11]
[6.89898004,7.29765392]	[6.88299293,7.29882524]
[3.97569664,4.405301763]	[3.95697257,4.40666378]
[5.26906122,5.656550266]	[5.25126311,5.65782779]
[2.04981395,2.3273262467]	[2.03528617,2.32835018]
[1.00461178,1.1871714692]	[0.99390173,1.18791644]

6. Conclusion

The problem of solving interval linear systems of equations is very important in practical applications.

A simple method for determining an outer solution to the linear system considered has been suggested in section 5 by using the method presented in section 4. Some numerical and practical examples are solved. The methods that presented can be applied to big real life problems such as structural engineering [18,23] without any problems.

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