On cubic Padé Approximation to the exponential function and its application in solving diffusion-convection equation

Jing-Hua Gao, Mei-Yan Lin
School of Science, Dalian Jiaotong University, Dalian, 116028, P. R. China

Abstract

Diagonal cubic Hermite-Padé approximation to the exponential function with coefficient polynomials of degree at most m is considered. Explicit formulas and differential equations are obtained for the coefficient polynomials. An exact asymptotic expression is obtained for the error function and it is also shown that these generalized Padé-type approximations can be used to asymptotically minimize the expressions on the unit disk. As an application, a class of local analytical difference schemes based on diagonal cubic Padé approximation for diffusion-convection equation with constant coefficients is proposed.

Keywords: *Padé-type approximant; Cubic Hermite-Padé approximation; Asymptotic formula; Differential equation*

1 Introduction

The Padé approximation theory has been widely used in problems of theoretical physics[1][3][4][5], numerical analysis[9][10], and electrical engineering, especially in modal analysis model[2], order reduction of multivariable systems[6][11][13][14].

We consider approximations of e^{-x} generated by finding polynomials P_m, Q_m, R_m and S_m so that

$$E_m(x) := P_m(x)e^{-3x}+Q_m(x)e^{-2x}+R_m(x)e^{-x}+S_m(x) = O(x^{4(m+1)-1}), \qquad (1)$$

where $P_m, Q_m, R_m, S_m \in \pi_m$ (the vector space of all algebraic polynomials of degree at most m), and P_m has leading coefficient 1. The approximation of e^{-x} is given by one of the following three functions ($j = 0, 1, 2$)

$$\delta_{jm}(x) := \omega_1^j \sqrt[3]{-\frac{b}{2}+\sqrt{\left(\frac{b}{2}\right)^2+\left(\frac{a}{3}\right)^3}} + \omega_2^j \sqrt[3]{-\frac{b}{2}-\sqrt{\left(\frac{b}{2}\right)^2+\left(\frac{a}{3}\right)^3}} - \frac{Q_m}{3P_m},$$

which is real and closest to 0, where

$$a = \frac{3P_mR_m - Q_m^2}{3P_m^2},$$

$$b = \frac{2Q_m^3 - 9P_mQ_mR_m + 27P_m^2S_m}{27P_m^3};$$

$$\omega_1 = \frac{-1+\sqrt{3}i}{2},\ \omega_2 = \frac{-1-\sqrt{3}i}{2}.$$

Obviously, $\delta_{jm}(x)$ is the natural cubic generalization of the main diagonal Padé approximant $-\hat{Q}_m/\hat{P}_m$ satisfying

$$\hat{P}_m(x)e^{-x} + \hat{Q}_m(x) = O(x^{2(m+1)-1})$$

and the diagonal quadratic Hermite-Padé approximant [7] $\left(-q_m + \sqrt{q_m^2 - 4p_mr_m}\right)\Big/(2p_m)$ satisfying

$$p_m(x)e^{-2x} + q_m(x)e^{-x} + r_m(x) = O(x^{3(m+1)-1}). \qquad (2)$$

Our primary aim is to derive the exact asymptotic formula for $\{E_m\}$, the explicit formulae of $\{P_m\}, \{Q_m\}, \{R_m\}, \{S_m\}, \{E_m\}$ and to treat some minimization problems concerning related approximations on the unit disk in **C**.

Exact results concerning best rational approximation to the exponential function, particularly the Meinardus conjecture, have attracted much attention ([8]). Theorem 4 can be viewed as a cubic version of this conjecture on the disk. A linear version, due to Trefethen appeared in [15]; a quadratic version on the disk given by Borwein can be found in [7].

As an application, this paper proposes a class of local analytical difference schemes based on cubic Padé approximation to e^{-x} for the following diffusion-convection equation

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + \varepsilon\frac{\partial^2 u}{\partial x^2} \qquad (3)$$

with the initial condition $u(x,0) = \phi(x),\ 0 \le x \le 1$; and the boundary condition $u(0,t) = \alpha,\ u(1,t) = \beta$; where $\varepsilon > 0,\ a \ne 0,\ \alpha,\ \beta$ are all real constants.

2 Explicit Formulae of the Polynomial Coefficients

Let

$$p_m(x) = m!\sum_{j=0}^{m}\frac{c_j x^j}{j!},$$

$$q_m(x) = -2^{m+1}m!\sum_{j=0}^{m}\frac{d_j x^j}{j!},$$

$$r_m(x) = (-1)^m p_m(-x),$$

with

$$c_j = \sum_{k=0}^{m-j}\frac{1}{2^k}\binom{2m-(k+j)}{m}\binom{m+k}{m};$$

$$d_j = \sum_{k=0}^{m-j}(-1)^{k-j}\binom{2m-(k+j)}{m}\binom{m+k}{m}.$$

Then $\left(-q_m + \sqrt{q_m^2 - 4p_m r_m}\right)\Big/(2p_m)$ is the diagonal quadratic Hermite-Padé approximant [7] to e^{-x}. Also, p_m, q_m and r_m satisfy (2) and

$$\begin{aligned} & p_m(x)e^{-2x} + q_m(x)e^{-x} + r_m(x) \\ &= \frac{2^{m+1}x^{3m+2}}{m!m!}\int_0^1\int_0^1 (1-u)^m u^m \\ &\times e^{-uvx}(1-v)^m v^{2m+1} e^{-vx}\mathrm{d}u\mathrm{d}v. \qquad (4)\end{aligned}$$

Now let

$$P_m(x) := \frac{e^{3x}3^{m+1}}{m!}\int_x^\infty (t-x)^m p_m(t)e^{-3t}\mathrm{d}t. \qquad (5)$$

Then

$$\begin{aligned} P_m(x) &= \frac{e^{3x}3^{m+1}}{m!}\int_x^\infty (t-x)^m e^{-3t} m!\sum_{j=0}^{m}\frac{c_j t^j}{j!}\mathrm{d}t \\ &= m!\sum_{i=0}^{m}\sum_{j=i}^{m}\frac{c_j}{3^{j-i}}\binom{m+j-i}{m}\frac{x^i}{i!}.\end{aligned}$$

If we set

$$a_i := \sum_{j=i}^{m} 3^{i-j}c_j\binom{m+j-i}{m},$$

then

$$P_m(x) = m!\sum_{i=0}^{m}\frac{a_i x^i}{i!}. \qquad (6)$$

Note that P_m is a polynomial of degree m with highest coefficient 1.

Let

$$Q_m(x) = \frac{e^{2x}3^{m+1}}{m!}\int_x^\infty (t-x)^m q_m(t)e^{-2t}\mathrm{d}t \qquad (7)$$

$$:= -3^{m+1}m!\sum_{i=0}^{m}\frac{b_i x^i}{i!}, \qquad (8)$$

with

$$b_i = \sum_{j=i}^{m} d_j 2^{i-j}\binom{m+j-i}{m}.$$

Then $Q_m(x)$ is a polynomial with integer coefficients.

Let

$$R_m(x) := \frac{e^{x}3^{m+1}}{m!}\int_x^\infty (t-x)^m r_m(t)e^{-t}\mathrm{d}t \qquad (9)$$

$$= (-1)^m 3^{m+1}m!\sum_{i=0}^{m}\frac{e_i x^i}{i!}, \qquad (10)$$

where

$$e_i := \sum_{j=i}^{m} c_j(-1)^j\binom{m+j-i}{m},$$

then $R_m(x)$ is a polynomial with integer coefficients.

Define S_m by

$$\begin{aligned} S_m(x) &= -\frac{3^{m+1}}{m!}\int_0^\infty (t-x)^m e^{-t} \\ &\times\left\{p_m(t)e^{-2t} + q_m(t)e^{-t} + r_m(t)\right\}\mathrm{d}t,\end{aligned}$$

then

$$S_m(x) = m!\sum_{i=0}^{m}\frac{s_i x^i}{i!}, \qquad (11)$$

with

$$\begin{aligned} s_i &= (-1)^{i+1}\sum_{j=0}^{m-i}\binom{2m-j-i}{m} \\ &\times\sum_{k=0}^{j}\binom{m+k}{m}\binom{m+j-k}{m}\frac{1}{3^k 2^{j-k}}.\end{aligned}$$

Finally, let

$$\begin{aligned} E_m(x) &= -\frac{3^{m+1}}{m!}\int_0^x (t-x)^m e^{-t} \\ &\cdot\frac{2^{m+1}t^{3m+2}}{m!m!}\int_0^1\int_0^1 u^m(1-u)^m\end{aligned}$$

$$
\begin{aligned}
&\cdot e^{-uvt}(1-v)^m v^{2m+1} e^{-vt}\mathrm{d}u\mathrm{d}v\mathrm{d}t \qquad (12)\\
=\ & (-1)^{m+1}\frac{6^{m+1}x^{4m+3}}{m!m!m!}\\
&\cdot\int_0^1\int_0^1\int_0^1 (1-u)^m u^m e^{-uvwx}\\
&\cdot(1-v)^m v^{2m+1} e^{-vwx} w^{3m+2}\\
&\cdot(1-w)^m e^{-wx}\mathrm{d}u\mathrm{d}v\mathrm{d}w\ (t=wx). \qquad (13)
\end{aligned}
$$

Now we may establish the basic theorem.

Theorem 1.

$$
\begin{aligned}
E_m(x) &:= P_m(x)e^{-3x}+Q_m(x)e^{-2x}+R_m(x)e^{-x}+S_m(x)\\
&= O(x^{4(m+1)-1}),
\end{aligned}
$$

where E_m, P_m, Q_m, R_m and S_m are given by (12), (6), (8), (10) and (11) respectively.

Proof By (12) and (4)

$$
\begin{aligned}
&E_m(x)\\
=\ & -\frac{3^{m+1}}{m!}\int_0^x (t-x)^m e^{-t}\\
&\cdot\left\{p_m(t)e^{-2t}+q_m(t)e^{-t}+r_m(t)\right\}\mathrm{d}t\\
=\ & \frac{3^{m+1}}{m!}\int_x^\infty (t-x)^m p_m(t)e^{-3t}\mathrm{d}t\\
&+\frac{3^{m+1}}{m!}\int_x^\infty (t-x)^m q_m(t)e^{-2t}\mathrm{d}t\\
&+\frac{3^{m+1}}{m!}\int_x^\infty (t-x)^m r_m(t)e^{-t}\mathrm{d}t\\
&-\frac{3^{m+1}}{m!}\int_0^\infty e^{-t}(t-x)^m\\
&\cdot\left\{p_m(t)e^{-2t}+q_m(t)e^{-t}+r_m(t)\right\}\mathrm{d}t\\
=\ & P_m(x)e^{-3x}+Q_m(x)e^{-2x}+R_m(x)e^{-x}+S_m(x).
\end{aligned}
$$

With (13), the theorem has been proved.

3 Asymptotics and Exact Minimization

We now turn to asymptotic estimate for $\{E_m\}$. As usual, $i_m \sim j_m$ means $i_m/j_m \to 1$ $(m\to\infty)$. Throughout this paper the asymptotics are concerning the variable m.

In order to give the asymptotic, we need the following lemma.

Lemma 2 [15].

$$
\begin{aligned}
&\int_0^1 (1-t)^{\alpha m}t^{\beta m}e^{-\gamma t}\mathrm{d}t\\
\sim\ & e^{-\frac{\beta\gamma}{\alpha+\beta}}\int_0^1 (1-t)^{\alpha m}t^{\beta m}\mathrm{d}t\\
=\ & e^{-\frac{\beta\gamma}{\alpha+\beta}}\frac{(\alpha m)!(\beta m)!}{((\alpha+\beta)m+1)!},
\end{aligned}
$$

where α, β are positive numbers, $\alpha m, \beta m$ are positive integers.

Now we can give the asymptotic estimates of $\{E_m\}$.

Theorem 3.

$$
E_m(x)\sim\frac{(-6)^{m+1}m!}{(4m+3)!}x^{4m+3}e^{-\frac{3}{2}x}.
$$

The asymptotic is uniformly on the bounded subsets of $\mathbf{C}$.

Proof From (13)

$$
\begin{aligned}
E_m(x)\ &\sim\ \frac{(-6)^{m+1}x^{4m+3}}{(2m+1)!m!}\int_0^1\int_0^1 (1-v)^m\\
&\cdot e^{-\frac{3}{2}vwx}v^{2m+1}(1-w)^m w^{3m+2}e^{-wx}\mathrm{d}v\mathrm{d}w\\
&\sim\ \frac{(-6)^{m+1}x^{4m+3}}{(3m+2)!}\int_0^1 (1-w)^m\\
&\cdot w^{3m+2}e^{-\frac{12m+5}{2(3m+1)}wx}\mathrm{d}w\\
&\sim\ \frac{(-6)^{m+1}m!}{(4m+3)!}x^{4m+3}e^{-\frac{(3m+2)(12m+5)}{2(4m+3)(3m+1)}x}\\
&\sim\ \frac{(-6)^{m+1}m!}{(4m+3)!}x^{4m+3}e^{-\frac{3}{2}x}.
\end{aligned}
$$

Both of the first two asymptotics follow from the elementary relation of Lemma 2. It is easy to check directly from (12) that $E_m(x)/\{(-6)^{m+1}m!x^{4m+3}e^{-\frac{3}{2}x}/(4m+3)!\}$ is uniformly bounded on a compact subsets of $\mathbf{C}$. The uniformity of the asymptotic now follows from Vitali's theorem [7].

We wish to uniformly minimize over $D := \{z \in \mathbf{C} : |z|\le 1\}$,

$$
w_m(z) := s_m(z)e^{-3z}+t_m(z)e^{-2z}+u_m(z)e^{-z}+v_m(z),
$$

where $s_m, t_m, u_m, v_m \in \pi_m$ and s_m has highest coefficient 1.

Firstly, we have

Theorem 4. For $|z|=1$,

$$
\left|E_m\left(z+\frac{3}{2(4m+3)}\right)\right|\sim\frac{6^{m+1}m!}{(4m+3)!}.
$$

Proof This follows from Theorem 2 and the observation that

$$
\left(z+\frac{3}{2(4m+3)}\right)^{4m+3}\sim z^{4m+3}e^{\frac{3}{2z}},
$$

and the fact that for $|z|=1$, $|e^{\frac{3}{2}(\frac{1}{z}-z)}|=1$.

Let

$$
P_m^*(z)=P_m\left(z+\frac{3}{2(4m+3)}\right),
$$

$$
Q_m^*(z)=Q_m\left(z+\frac{3}{2(4m+3)}\right),
$$

$$R^*_m(z) = R_m\left(z + \frac{3}{2(4m+3)}\right),$$

$$S^*_m(z) = S_m\left(z + \frac{3}{2(4m+3)}\right).$$

Let $||\cdot||_D$ denote the supremum norm on D.

Now we need the following lemma.

Lemma 5 [12] Suppose $m_1, m_2, \cdots, m_l$ are positive integers such that $m_1 + m_2 + \cdots + m_l = n$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_l$ are real numbers. Let $f_1(z),\ f_2(z),\ \cdots,\ f_n(z)$ denote $e^{\lambda_1 z},\ ze^{\lambda_1 z},\ \cdots,\ z^{m_1-1}e^{\lambda_1 z};\ e^{\lambda_2 z},\ ze^{\lambda_2 z},\ \cdots,\ z^{m_2-1}e^{\lambda_2 z};\ \cdots,\ e^{\lambda_l z}, ze^{\lambda_l z},\ \cdots,\ z^{m_l-1}e^{\lambda_l z}$ respectively. If $|k_1| + |k_2| + \cdots + |k_{m_1}| > 0, \quad |k_{n-m_l+1}| + \cdots + |k_{n-1}| + |k_n| > 0$, and N denotes the zero number of function $k_1f_1(z) + k_2f_2(z) + \cdots + k_nf_n(z)$ in the region $\Omega = \{z : \xi \le Imz \le \eta\}$. Then

$$\frac{(\lambda_l - \lambda_1)(\eta-\xi)}{2\pi} - n + 1 \le N \le \frac{(\lambda_l - \lambda_1)(\eta-\xi)}{2\pi} + n - 1.$$

By this lemma, we can prove the main result of this section.

Theorem 6. (a)

$$||P^*_m(z)e^{-3z} + Q^*_m(z)e^{-2z} + R^*_m(z)e^{-z} + S^*_m(z)||_D \sim \frac{6^{m+1}m!}{(4m+3)!}.$$

(b) *Let*

$$w^*_m = \min_{s,t,u,v\in\pi_m} ||s_m e^{-3z} + t_m e^{-2z} + u_m e^{-z} + v_m||_D,$$

where $s = z^m + \cdots$. *Then*

$$w^*_m \sim \frac{6^{m+1}m!}{(4m+3)!}.$$

Proof Part (a) is just a restatement of Theorem 4. Observe that P^*_m has leading coefficient 1.
To prove part (b) we use the fact that a nonzero expression of the form

$$h_1(z)e^{-3z} + h_2(z)e^{-2z} + h_3(z)e^{-z} + h_4(z),$$

where h_1, h_2, h_3, h_4 are polynomials, the sum of whose degrees is k, can have at most $k+3$ zeros in D. This is winding number argument and is proved in Lemma 5 with $l = 4,\ \xi = -1,\ \eta = 1,\ \lambda_1 = -3,\ \lambda_2 = -2,\ \lambda_3 = -1,\ \lambda_4 = 0$ and $n = k + 4$. Thus

$$w^*_m \ge \min_{|z|=1} |P^*_m(z)e^{-3z} + Q^*_m(z)e^{-2z} + R^*_m(z)e^{-z} + S^*_m(z)|.$$

If this were not the case we could find $s, t, u, v \in \pi_m$ with s having leading coefficient 1 so that, for $|z| = 1$

$$|se^{-3z} + te^{-2z} + ue^{-z} + v| < |P^*_m e^{-3z} + Q^*_m e^{-2z} + R^*_m e^{-z} + S^*_m|.$$

By Rouché's theorem this would imply that

$$(s - P^*_m)e^{-3z} + (t - Q^*_m)e^{-2z} + (u - R^*_m)e^{-z} + (v - S^*_m) \tag{14}$$

has at least $4m + 3$ zeros in D. However, since $s - P^*_m$ has degree at most $m - 1$, the sum of the degrees of the coefficients in (14) is at most $4m - 1$, and we have contradicted the above result from Lemma 5.

We note that

$$||P_m e^{-3z} + Q_m e^{-2z} + R_m e^{-z} + S_m||_D \sim e^{\frac{3}{2}} \frac{6^{m+1}m!}{(4m+3)!}$$

and so, up to a small constant, the cubic Hermite-Padé approximant is optimal in the sense of Theorem 6. The trick of shifting the center of the approximation to make the error curve have asymptotically constant norm on D is due to Braess [8] who used it to get the right constant in the Meinardus' conjecture.

It is easy to show that the coefficient polynomials of the diagonal cubic Hermite-Padé approximant are linked by the following fourth-order differential equations.

Theorem 7.

$$\begin{aligned}
&\text{(a)} & -6mP_{m-1}(x) &= P_m^{''''} - 6P_m^{'''} + 11P_m^{''} - 6P_m^{'}.\\
&\text{(b)} & -6mQ_{m-1}(x) &= Q_m^{''''} - 2Q_m^{'''} - Q_m^{''} + 2Q_m^{'}.\\
&\text{(c)} & -6mR_{m-1}(x) &= R_m^{''''} + 2R_m^{'''} - R_m^{''} - 2R_m^{'}.\\
&\text{(d)} & -6mS_{m-1}(x) &= S_m^{''''} + 6S_m^{'''} + 11S_m^{''} + 6S_m^{'}.
\end{aligned}$$

4 Local analytical difference schemes based on cubic Padé approximation

In order to investigate the application of the diagonal cubic Padé approximation to e^{-x}, we discuss the problem (1.6).

We rewrite the above equation (1.6) as

$$au_x + \varepsilon u_{xx} = c. \tag{15}$$

In the case that c is a constant, we get the solution of (15) after integration:

$$u = c_1 e^{\lambda x} + c_2 + \frac{c}{a}x, \quad \lambda = -\frac{a}{\varepsilon}.$$

After discretization, we have

$$u_i = \frac{u_{i+1} + e^{\lambda h}u_{i-1}}{1 + e^{\lambda h}} + c\frac{h(e^{\lambda h} - 1)}{a(1 + e^{\lambda h})},$$

where $h = \frac{1}{N}$ is the common step width. Therefore

$$c = \frac{a\left(u_{i+1} - (1+e^{\lambda h})u_i + e^{\lambda h}u_{i-1}\right)}{h(1-e^{\lambda h})},$$

$$i = 1, 2, ..., N-1.$$

Now letting $c = \frac{\mathrm{d}u_i}{\mathrm{d}t}$, we have

$$\left\{\begin{array}{lcl} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} & = & -A\mathbf{u} + \mathbf{b}, \\ \mathbf{u}(0) & = & \mathbf{\Phi}, \end{array}\right. \tag{16}$$

where

$$\mathbf{u} = (u_1, u_2, ..., u_{N-1})^T,$$

$$\mathbf{b} = \left(\frac{a\alpha e^{\lambda h}}{h(1-e^{\lambda h})}, 0, ..., 0, \frac{\alpha\beta}{h(1-e^{\lambda h})}\right)^T,$$

$\mathbf{\Phi} = (\phi(x_1), \phi(x_2), ..., \phi(x_{N-1}))^T$ are all $(N-1)$-dimensional column vectors,

$$A = \frac{a}{h(1-e^{\lambda h})}$$

$$\cdot\begin{pmatrix} 1+e^{\lambda h} & -1 & 0 & \cdots & 0 \\ -e^{\lambda h} & 1+e^{\lambda h} & -1 & \ddots & \vdots \\ 0 & -e^{\lambda h} & 1+e^{\lambda h} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -e^{\lambda h} & 1+e^{\lambda h} \end{pmatrix}$$

is an $(N-1)\times(N-1)$ tridiagonal matrix.

So the solution of problem (16) is

$$\mathbf{u}(t) = e^{-At}(\mathbf{\Phi} - A^{-1}\mathbf{b}) + A^{-1}\mathbf{b}, \tag{17}$$

where $e^{-At} = \sum_{n=0}^{\infty} \frac{1}{n!}(-At)^n$.

Equivalently, (17) can be written as the following forms:

$$\mathbf{u}(t+j\tau) = e^{-jA\tau}\{\mathbf{u}(t) - A^{-1}\mathbf{b}\} + A^{-1}\mathbf{b} \ \ (j = 1, 2, 3), \tag{18}$$

where $\tau > 0$ is the increment.

Denoting

$$T_m(x) = -\frac{Q_m(x)}{P_m(x)}, \ \ U_m(x) = -\frac{R_m(x)}{P_m(x)},$$

$$V_m(x) = -\frac{S_m(x)}{P_m(x)},$$

we have

$$e^{-3x} = T_m(x)e^{-2x} + U_m(x)e^{-x} + V_m(x) + O(x^{4m+3}).$$

Replacing $e^{-3A\tau}$ in (18) with $T_m(A\tau)e^{-2A\tau} + U_m(A\tau)e^{-A\tau} + V_m(A\tau)$, we get a class of 3-step local analytical difference schemes based on diagonal cubic Padé approximation:

$$\begin{aligned} \mathbf{u}(t+3\tau) &= T_m(A\tau)\mathbf{u}(t+2\tau) \\ &+ U_m(A\tau)\mathbf{u}(t+\tau) + V_m(A\tau)\mathbf{u}(t) \\ &+ \{I - T_m(A\tau) - U_m(A\tau) - V_m(A\tau)\}A^{-1}\mathbf{b}. \end{aligned} \tag{19}$$

According to (19), we can list three simple schemes as follows:

Scheme 1. $m = 0$, $P_m = 1$, $Q_m = -3$, $R_m = 3$, $S_m = -1$;

$$\mathbf{u}(t+3\tau) = 3\mathbf{u}(t+2\tau) - 3\mathbf{u}(t+\tau) + \mathbf{u}(t).$$

Scheme 2. $m = 1$,

$$P_m = x + \frac{11}{3}, \ Q_m = 9(x+1),$$

$$R_m = 9(x-1), \ S_m = x - \frac{11}{3};$$

$$\mathbf{u}(t+3\tau) = \left(\frac{11}{3}I + \tau A\right)^{-1}\Big\{-9(I+\tau A)\mathbf{u}(t+2\tau)$$

$$+9(I-\tau A)\mathbf{u}(t+\tau) + \left(\frac{11}{3}I - \tau A\right)\mathbf{u}(t) + 20\tau\mathbf{b}\Big\};$$

where I is the $(N-1)\times(N-1)$ unit matrix.

Scheme 3. $m = 2$,

$$P_m = x^2 + 11x + \frac{103}{3}, \ Q_m = 27(x^2 + 3x + 9),$$

$$R_m = -27(x^2 - 3x + 9), \ S_m = x^2 - 11x + \frac{103}{3};$$

$$\mathbf{u}(t+3\tau) = \left(\frac{103}{3}I + 11\tau A + \tau^2 A^2\right)^{-1}$$

$$\cdot\Big\{27(9I + 3\tau A + \tau^2 A^2)\mathbf{u}(t+2\tau)$$

$$-27(9I - 3\tau A + \tau^2 A^2)\mathbf{u}(t+\tau)$$

$$+\Big(\frac{103}{3}I - 11\tau A + \tau^2 A^2\Big)\mathbf{u}(t) - 140\tau\mathbf{b}\Big\}.$$

References

[1] G.A. Baker, Essentials of Padé Approximants, New York: Academic Press, 1975.

[2] P. Barone, R. March, Some properties of the asymptotic location of poles of Padé approximants to noisy rational functions, relevant for modal analysis, IEEE Trans. Sign. Processing, SP-**46**(1998) 2448–2457.

[3] D. Belkić , Strikingly stable convergence of the Fast Padé Transform (FPT) for high-resolution parametric and non-parametric signal processing of Lorentzian and non-Lorentzian spectra , Nuclear Instruments and Methods in Physics Research, A **525**(2004) 366-371.

[4] D. Belkić , Analytical continuation by numerical means in spectral analysis using the Fast Padé Transform (FPT), Nuclear Instruments and Methods in Physics Research, A **525**(2004) 372-378.

[5] D. Belkić , Error analysis through residual frequency spectra in the fast Padé transform (FPT), Nuclear Instruments and Methods in Physics Research, A **525**(2004) 379-386.

[6] Y. Bistrits, U. Shaked, Discrete multivariable system approximations by minimal Padé-type stable models, IEEE Trans. Circuit Syst. , CAS-**31**(1984) 382-390.

[7] P.B. Borwein, Quadratic Hermite-Padé approximation to the exponential function, Constr. Approx., **2**(1986) 291-302.

[8] D. Braess, On the conjecture of Meinardus on rational approximation of e^x, II, J.Approx.Theory, **40**(1984) 375-379.

[9] E. Çelik, On the numerical solution of chemical differential-algebraic equations by Pade series, Appl. Math. Comput. **153**(2004) 13-17.

[10] W. B. Gragg, The Padé table and its relation to certain algorithms of numerical analysis, SIAM Rev. , **14**(1972) 1-62.

[11] P. N. Paraskevopoulos, Padé-type order reduction of two-dimensional systems, IEEE Trans. Circuit Syst. , CAS-**27**(1980) 413-416.

[12] G. Pólya, G. Szegö, Problems and Thoerems in Analysis, I, Berlin: Springer-Verlag, 1972.

[13] Y. Shamash, Multivariable system reduction via modal methods and Padé approximation, IEEE Trans. Automat. Contr., AC-**20**(1975) 815-817.

[14] Y. Shamash, Linear system reduction using Padé approximation to allow retention of dominant modes, Int. J. Contr. , **21**(1975) 257-272.

[15] L.N. Trefethen, The asymptotic accuracy of rational best approximation to e^z on a disk, J. Approx. Theory, **40**(1984) 380-383.

Jing-Hua Gao was born in Heilongjiang, China, in 1962. She received the Bachelor Degree in mathematics from Harbin Normal University of China in 1983. She joined the Department of Mathematics, Dalian Jiaotong University, China in 2003. She is now an associate professor. Her research interests include numerical approximation and its application.

Mei-Yan Lin was born in Liaoning, China, in 1973. She received the Master Degree in mathematics from Xi'an University of Science and Technology of China in 2003, and then joined the Department of Mathematics, Dalian Jiaotong University, China. Her research interests include numerical approximation and its application.