New robust stability criteria of neutral-type neural networks with interval time-varying delays

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Abstract

The existence, uniqueness and global robust exponential stability is analyzed for the equilibrium point of a class of neutral-type neural networks with time-varying delays. By dividing the variation interval of the time delay into two subintervals with equal length, a more general type of Lyapunov functionals is defined. Following the idea of convex combination and free-weighting matrices method, new delay-dependent stability criteria are presented in terms of linear matrix inequalities (LMIs). Three examples are also given to illustrate the effectiveness and less conservativeness of our proposed conditions than some previous ones.

Keywords: *Global robust exponential stability, neutral-type neural networks, Jensen integral inequality, linear matrix inequality(LMI), free-weighting matrix*

1. Introduction

Recurrent Neural Networks can be represented as differential equations that describe the evolution of the model as functions of time. These differential equations have received increasing interest due to their promising potential applications in areas such as classification, combinatorial optimization, parallel computing, signal processing and pattern recognition. Such applications heavily depend on the dynamic behavior of networks, therefore, the analysis of these dynamic behaviors is a necessary step for practical design of neural networks. Up to now, many important results on the stability have been reported in the literature, see e.g. [2,5–23] and references therein.

In the design of neural networks, however, one is not only interested in global stability, but also in some other performances. Particularly, it is often desirable to have a neural network that converges fast enough in order to achieve fast response. Considering this, many researchers have studied the exponential stability analysis problem for delayed neural networks and a great number of results on this topic have been reported in the literature [2,6,8,9,12, 13,16–20].

On the other hand, due to the complicated dynamic properties of the neural cells in the real world, the existing neural networks model in many cases can't characterize the properties of a neural reaction process precisely, so the neural networks system will contain some information about the derivative of the past state to model the dynamics for such complex neural reactions [2,8–11,18,22]. For neutral-type neural networks with constant delays, by employing LMI techniques, Xu et al. [18] obtained a delay-dependent exponential stability condition with the assumption of the boundedness and monotonic non-decreasing of the activation functions; by adopting the parameter model transform method, using free weighting matrices approach and LMI techniques, Park J.H. [10] established a delay-dependent stability condition also with the assumption of the boundedness and monotonic non-decreasing of the activation functions; by using semi-free weighting matrices approach and LMI techniques, Mai et al. [9] derived two delay-dependent exponential stability criterion with the assumption of the boundedness of the activation functions. For neutral-type neural networks with time-varying delays, using inequalities of vector and norm, employing Razumikhin's method, Cao et al. [2] achieved two stability conditions; by using Jensen integral inequality, LMIs and Razumikhin-like approaches, Lien et al. [8] proposed several delay-dependent and delay-independent stability criteria with multiple delays; by adopting free weighting matrices approach and LMI techniques, Park J.H. [10] established a delay-dependent stability condition with the assumption of the boundedness and monotonic non-decreasing of the activation functions; by employing LMI techniques, Zhu et al. [22] derived two delay-dependent robust stability criteria with the assumption of the boundedness of the activation functions. However, to the best of our knowledge, there are no results proving the existence of the equilibrium point of such neutral-type neural networks up to now, especially with unbounded activation functions.

In this paper, we consider the existence, uniqueness and global robust exponential stability of the uncertain neutral-type neural networks with time-varying delays in this paper. By dividing the variation interval of the time delay into two subintervals with equal length [21], we construct a new Lyapunov-Krasovskii functional and derive new sufficient conditions, which are delay-dependent and computa-tionally efficient. Following the idea of convex combi-nation [23], less conservative results are obtained by using the free-weighting matrix approach [16] and Jensen integral inequality. Finally, three illustrative examples are also given to demonstrate the effectiveness of the proposed results.

2. Problem description

Considering the following neutral type neural networks with interval time-varying delays:

$$\dot{x}(t) = -\bar{C}x(t) + \bar{A}\tilde{f}(x(t)) + \bar{B}\tilde{g}(x(t-\tau(t))) + \bar{E}\dot{x}(t-\sigma(t)) + J, \quad (1)$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$ is the neural state vector, $x(t-\tau(t)) = (x_1(t-\tau_1(t)), x_2(t-\tau_2(t)), ..., x_n(t-\tau_n(t)))^T$, $\bar{C} = C + \Delta C(t)$, $\bar{A} = A + \Delta A(t)$, $\bar{B} = B + \Delta B(t)$, $\bar{E} = E + \Delta E(t)$. $C = \mathrm{diag}\{c_1, c_2, ..., c_n\}$ is a positive diagonal matrix, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $E = (e_{ij})_{n\times n}$ are known constant matrices, $\Delta C(t), \Delta A(t), \Delta B(t), \Delta E(t)$ are parametric uncertainties, $0 \le h_0 \le \tau_i(t) \le h_0 + 2h (i = 1, ..., n), 0 \le \sigma(t) \le d$ are the time-varying delays, where h_0, h, d are non-negative constants. J is the constant external input vector, and $\tilde{f}(x(t)) = \left(\tilde{f}_1(x_1(t)), \tilde{f}_2(x_2(t)), ..., \tilde{f}_n(x_n(t))\right)^T$, $\tilde{g}(x(t-\tau(t))) = \left(\tilde{g}_1(x_1(t-\tau_1(t))), \tilde{g}_2(x_2(t-\tau_2(t))), ..., \tilde{g}_n(x_n(t-\tau_n(t)))\right)^T \in \mathbb{R}^n$ denote the neural activation functions. It is assumed that $\tilde{f}_j(s), \tilde{g}_j(s)$ satisfy the following condition:

Assumption 1 There exist constant scalars l_{ij} such that $l_{1j} < l_{2j}, l_{3j} < l_{4j}$ and

$$l_{1j} \le \frac{\tilde{f}_j(s_1) - \tilde{f}_j(s_2)}{s_1 - s_2} \le l_{2j}, \quad (2)$$

$$l_{3j} \le \frac{\tilde{g}_j(s_1) - \tilde{g}_j(s_2)}{s_1 - s_2} \le l_{4j}, \quad (3)$$

for any $s_1, s_2 \in \mathrm{R}$, $s_1 \ne s_2$, $j = 1, 2, ..., n$.

Moreover, we assume that the initial condition of system (1) has the form

$$x_i(t) = \phi_i(t), \quad t \in [-\bar{h}, 0]$$

where $\phi_i(t)(i = 1, 2, ..., n)$ are continuous functions, $\bar{h} = \max\{h_0 + 2h, d\}$.

Throughout this paper, let $\|y\|$ denotes the Euclidean norm of a vector $y \in \mathbb{R}^n, W^T, W^{-1}, \lambda_M(W), \lambda_m(W)$ and $\|W\| = \sqrt{\lambda_M(W^T W)}$ denote the transpose, the inverse, the largest eigenvalue, the smallest eigenvalue, and the spectral norm of a square matrix W, respectively. Let W>0(<0) denote a positive (negative) definite symmetric matrix, I denote an identity matrix with compatible dimension.

The definition of exponential stability is now given.

Definition 1 ([18]) The system (1) is said to be globally exponentially stable if there exist constants $r \ge 0$ and M>1 such that

$$\|x(t)\| \le M\left(\sup_{-\bar{h}\le\theta\le 0}\{\|x(\theta), \|\dot{x}(\theta)\|\}\right)e^{-kt},$$

where k is called the exponential convergence rate.

The time-varying uncertain matrices $\Delta C(t), \Delta A(t), \Delta B(t), \Delta E(t)$ are defined by:

$$[\Delta C(t), \Delta A(t), \Delta B(t), \Delta E(t)] = H_0 E_0(t)[G_0, G_1, G_2, G_3], \quad (4)$$

where H_0, G_0, G_1, G_2, G_3 are known real constant matrices with appropriate dimensions. $E_0(t)$ is an unknown time-varying matrix satisfying

$$E_0^T(t)E_0(t) \le I. \quad (5)$$

In order to obtain the results, we need the following lemmas.

Lemma 1 (see [1]) Let H, K and L be real matrices of appropriate dimensions with $\mathrm{K} > 0$. Then for any vectors x and y with appropriate dimensions, the following matrix inequality holds:

$$2x^T \mathrm{H}\,\mathrm{L}y \le x^T \mathrm{H}\,\mathrm{K}^{-1}\mathrm{H}^T x + y^T \mathrm{L}^T \mathrm{K}\,\mathrm{L}y.$$

Lemma 2 (see [3]) Continuous map $T(x): \mathbb{R}^n \to \mathbb{R}^n$ is homeomorphic if T(x) is injective and $\lim_{\|x\|\to\infty} \|T(x)\| = \infty$.

Lemma 3 (see [20]) Assuming that function $g_j(s)$ is defined such that $0 \le g_j(s)/s \le \rho_j$, where $\rho_j > 0$, then the following inequality holds

$$\int_{\varsigma}^{\xi} (g_j(s) - g_j(\varsigma))\mathrm{d}s \le (\xi - \varsigma)(g_j(\xi) - g_j(\varsigma)).$$

Lemma 4 (see [4]) For any positive symmetric constant matrix $M \in \mathbb{R}^{n\times n}$, scalars $r_1 < r_2$ and vector function $\omega: [r_1, r_2] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then

$$\left(\int_{r_1}^{r_2} \omega(s)\mathrm{d}s\right)^T M \left(\int_{r_1}^{r_2} \omega(s)\mathrm{d}s\right) \le (r_2 - r_1)\int_{r_1}^{r_2} \omega^T(s) M \omega(s)\mathrm{d}s.$$

From Lemma 1, it is easy to see that the following Lemma holds

Lemma 5 (see [1]) Given matrices H,G with compatible dimensions, the following matrix inequality

$$HF(t)G+G^TF^T(t)H^T \le HQH^T+G^TQ^{-1}G$$

holds for any definite positive symmetric matrix Q, where matrix F(t) satisfies $F^T(t)F(t)\le I.$.

3. Existence and uniqueness of neutral-type neural networks

In order to study the existence and uniqueness of the equilibrium point, we define maps $\mathrm{H}(x)$ and $\mathrm{H}^*(x)$ respectively as follows:

$$\mathrm{H}(x)=-Cx+A\tilde{f}(x)+B\tilde{g}(x)+J$$

and $\mathrm{H}^*(x)=\mathrm{H}(x)-\mathrm{H}(0).$

Firstly, we present a delay-dependent criterion for the existence and uniqueness of system (1) with time delays $\tau_i(t)=\tilde{\tau}(t)(i=1,\dots,n),$ where $0\le h_0\le\tilde{\tau}(t)\le h_0+2h.$

Theorem 1. Under Assumptions 1, given constant scalars $r\ge 0, h_0\ge 0, h>0, d\ge 0, 0\le\eta<1, \varsigma<1,$ neural networks (1) has a unique equilibrium point for $0<h_0\le\tilde{\tau}(t)\le h_0+2h,$ $0\le\sigma(t)\le d,\ 0\le\dot{\tilde{\tau}}(t)\le\eta<1, \dot{\sigma}(t)\le\varsigma<1,$ if there exist constant scalar $\varepsilon>0,$ positive definite symmetric matrices $P=[P_{ij}]_{3\times3}, Q=[Q_{ij}]_{2\times2}, R=[R_{ij}]_{3\times3}, U=[U_{ij}]_{2\times2}, S_l$ $(l=1,\dots,6),$ positive diagonal matrices $T_m(m=1,2,3),$ $D_k=\mathrm{diag}\{d_{k1},d_{k2},\dots,d_{kn}\},$ real matrices X_0, X_k, Y_k $(k=1,2,3,4)$ with compatible dimensions such that the following LMIs hold $(i,\ j=1,2)$:

$$\begin{bmatrix}\Omega+\Sigma_i+\varepsilon\Theta^T\Theta & \Gamma_{ij}^T & \Psi^T H_0\\ \Gamma_{ij} & -e^{-2r(h_0+ih)}S_i & 0\\ H_0^T\Psi & 0 & -\varepsilon I\end{bmatrix}<0, \qquad (6)$$

where

$\Omega=[\ \Omega_{ij}\ ]_{13\times13}, \Sigma_1=[\ \Sigma_{ij}\ ]_{13\times13}, \Sigma_2=[\ \tilde{\Sigma}_{ij}\ ]_{13\times13},$

$\Omega_{11}=2rP_{11}-\mathrm{F}_1C-C\mathrm{F}_1^T+R_{11}+U_{11}+S_3-2L_1T_1L_2-2L_3T_2L_4$
$-e^{-2rh_0}S_4+4r(L_2-L_1)(D_1+D_2)+4r(L_4-L_3)(D_3+D_4)-e^{-2rd}S_6,$

$\Omega_{12}=(2rI-C)P_{12},\ \ \Omega_{13}=e^{-2rh_0}S_4,\ \ \Omega_{15}=(2rI-C)P_{13},$

$\Omega_{16}=P_{12},\ \Omega_{17}=P_{13},\ \ \Omega_{18}=e^{-2rd}S_6,\ \ \Omega_{19}=\mathrm{F}_1E,$

$\Omega_{1,10}=\mathrm{F}_1A-C(D_1-D_2)+(L_1+L_2)T_1,$

$\Omega_{1,11}=R_{12}-C\mathrm{F}_2^T+(L_3+L_4)T_2,\ \ \Omega_{1,12}=\mathrm{F}_1B,\ \ \Omega_{1,13}=-CX_0,$

$\Omega_{22}=2rP_{22}-(1-\eta)e^{-2r(h_0+2h)}R_{11}-2L_3T_3L_4,\ \Omega_{25}=2rP_{23},$

$\Omega_{26}=P_{22}-e^{-2r(h_0+2h)}R_{13},\ \Omega_{27}=P_{23},\ \Omega_{29}=P_{12}^TE,\ \Omega_{2,10}=P_{12}^TA,$

$\Omega_{2,12}=P_{12}^TB-(1-\eta)e^{-2r(h_0+2h)}R_{12}+(L_3+L_4)T_3,$

$\Omega_{33}=e^{-2rh_0}Q_{11}-e^{-2rh_0}S_3-e^{-2rh_0}S_4,\ \Omega_{34}=e^{-2rh_0}Q_{12},$

$\Omega_{44}=e^{-2rh_0}Q_{22}-e^{-2r(h_0+h)}Q_{11},\ \Omega_{45}=-e^{-2r(h_0+h)}Q_{12},$

$\Omega_{55}=2rP_{33}-e^{-2r(h_0+h)}Q_{22}-e^{-2r(h_0+2h)}U_{11},\ \Omega_{56}=P_{23}^T,$

$\Omega_{57}=P_{33}-e^{-2r(h_0+2h)}U_{12},\ \ \Omega_{59}=P_{13}^TE,\ \ \Omega_{5,10}=P_{13}^TA,\ \ \Omega_{5,12}=P_{13}^TB,$

$\Omega_{66}=-e^{-2r(h_0+2h)}R_{33},\ \Omega_{6,12}=-e^{-2r(h_0+2h)}R_{23}^T,\ \Omega_{77}=-e^{-2r(h_0+2h)}U_{22},$

$\Omega_{88}=-e^{-2rd}S_6,\ \Omega_{99}=-(1-\varsigma)e^{-2rd}S_5,\ \Omega_{9,10}=E^T(D_1-D_2),$

$\Omega_{9,11}=E^T\mathrm{F}_2^T,\ \ \Omega_{9,13}=-E^TX_0,\ \Omega_{10,10}=(D_1-D_2)A+A^T(D_1-D_2)-2T_1,$

$\Omega_{10,11}=A^T\mathrm{F}_2^T,\ \Omega_{10,12}=(D_1-D_2)B,\ \Omega_{10,13}=-A^TX_0,\ \Omega_{11,11}=R_{22}-2T_2,$

$\Omega_{11,12}=\mathrm{F}_2B,\ \Omega_{12,12}=-(1-\eta)e^{-2r(h_0+2h)}R_{22}-2T_3,\ \Omega_{12,13}=-B^TX_0,$

$\Omega_{13,13}=R_{33}+U_{22}+h^2(S_1+S_2)+h_0^2S_4+S_5+d^2S_6-X_0-X_0^T,$

$\Sigma_{22}=-X_1-X_1^T+X_3+X_3^T,\ \ \Sigma_{23}=X_1^T-X_2,\ \ \Sigma_{24}=-X_3^T+X_4,$

$\Sigma_{33}=X_2+X_2^T,\ \Sigma_{44}=-X_4-X_4^T-e^{-2r(h_0+2h)}S_2,\ \ \Sigma_{45}=e^{-2r(h_0+2h)}S_2,$

$\Sigma_{55}=-e^{-2r(h_0+2h)}S_2,\ \tilde{\Sigma}_{22}=-Y_1-Y_1^T+Y_3+Y_3^T,\ \tilde{\Sigma}_{24}=Y_1^T-Y_2,$

$\tilde{\Sigma}_{25}=-Y_3^T+Y_4,\ \tilde{\Sigma}_{33}=-e^{-2r(h_0+h)}S_1,\ \tilde{\Sigma}_{34}=e^{-2r(h_0+h)}S_1,$

$\tilde{\Sigma}_{44}=-e^{-2r(h_0+h)}S_1+Y_2+Y_2^T, \tilde{\Sigma}_{55}=-Y_4-Y_4^T,$

$\Theta=[-G_0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ G_3\ \ G_1\ \ 0\ \ G_2\ \ 0],$

$\Gamma_{11}=[0\ \ X_1\ \ X_2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0],$

$\Gamma_{12}=[0\ \ X_3\ \ 0\ \ X_4\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0],$

$\Gamma_{21}=[0\ \ Y_1\ \ 0\ \ Y_2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0],$

$\Gamma_{22}=[0\ \ Y_3\ \ 0\ \ 0\ \ Y_4\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0],$

$\Psi=[\ \mathrm{F}_1^T\ \ P_{12}\ \ 0\ \ 0\ \ P_{13}\ \ 0\ \ 0\ \ 0\ \ 0\ \ D_1-D_2\ \ \mathrm{F}_2^T\ \ 0\ \ 0\ ],$

$\mathrm{F}_1=P_{11}+R_{13}+U_{12}-D_1L_1+D_2L_2-D_3L_3+D_4L_4,$

$\mathrm{F}_2=R_{23}+D_3-D_4,\ \ L_i=\mathrm{diag}\{l_{i1},l_{i2},\dots,l_{in}\}, i=1,2,3,4,$

and other parameters $\Omega_{ij},\Sigma_{ij},\tilde{\Sigma}_{ij}(i\le j)$ are all equal to zero's, then system (1) has a unique equilibrium point.

Proof. By Schwarz inequality, Lemmas 1, 2 and Assumption 1 we can complete the proof, here is omitted.

Similar to Theorem 1, we obtain the following delay-dependent criterion for the existence and uniqueness of system (1) with different time delays $\tau_i(t)(i=1,\dots,n)$ satisfying $0\le h_0\le\tau_i(t)\le h_0+2h.$

Theorem 2. Under Assumption 1, given constant scalars $r\ge 0, h_0\ge 0, h>0, d\ge 0, 0\le\eta<1, \varsigma<1,$ neural networks (1) has a unique equilibrium point for $0<h_0$ $\le\tau_i(t)\le h_0+2h, 0\le\sigma(t)\le d, 0\le\dot{\tau}_i(t)(t)\le\eta<1, \dot{\sigma}(t)\le\varsigma<1,$ if there exist constant scalar $\varepsilon>0,$ positive definite symmetric matrices $P=[P_{ij}]_{3\times3}, Q=[Q_{ij}]_{2\times2}, U=[U_{ij}]_{2\times2},$ $S_l(l=1,\dots,6),$ positive diagonal matrices $R_m=\mathrm{diag}\{r_{m1},$ $r_{m2},\dots,r_{mn}\}, T_m(m=1,2,3), D_k,$ real matrices X_0, X_k, Y_k

$(k=1,2,3,4)$ with compatible dimensions such that the following LMIs hold (i,j=1,2):

$$\begin{bmatrix} \tilde{\Omega}+\Sigma_i+\varepsilon\Theta^T\Theta & \Gamma_{ij}^T & \tilde{\Psi}^T H_0 \\ \Gamma_{ij} & -e^{-2r(h_0+ih)}S_i & 0 \\ H_0^T\tilde{\Psi} & 0 & -\varepsilon I \end{bmatrix}<0,$$

where

$$\tilde{\Omega}=[\tilde{\Omega}_{ij}]_{13\times 13},$$

$$\tilde{\Omega}_{11}=2rP_{11}-F_3C-CF_3^T+R_1+U_{11}+S_3-2L_1T_1L_2-2L_3T_2L_4 -e^{-2rh_0}S_4+4r(L_2-L_1)(D_1+D_2)+4r(L_4-L_3)(D_3+D_4)-e^{-2rd}S_6,$$

$\tilde{\Omega}_{19}=F_3E,\ \tilde{\Omega}_{1,10}=F_3A-C(D_1-D_2)+(L_1+L_2)T_1,$

$\tilde{\Omega}_{1,11}=-C(D_3-D_4)+(L_3+L_4)T_2,\ \tilde{\Omega}_{1,12}=F_3B,$

$\tilde{\Omega}_{22}=2rP_{22}-(1-\eta)e^{-2r(h_0+2h)}R_1-2L_3T_3L_4,\ \tilde{\Omega}_{26}=P_{22},$

$\tilde{\Omega}_{2,12}=P_{12}^TB+(L_3+L_4)T_3,\ \tilde{\Omega}_{66}=-e^{-2r(h_0+2h)}R_3,\ \tilde{\Omega}_{9,11}=E^T(D_3-D_4),$

$\tilde{\Omega}_{10,11}=A^T(D_3-D_4),\ \tilde{\Omega}_{11,11}=R_2-2T_2,\ \tilde{\Omega}_{11,12}=(D_3-D_4)B,$

$\tilde{\Omega}_{12,12}=-(1-\eta)e^{-2r(h_0+2h)}R_2-2T_3,$

$\tilde{\Omega}_{13,13}=R_3+U_{22}+h^2(S_1+S_2)+h_0^2S_4+S_5+d^2S_6-X_0-X_0^T,$

$F_3=P_{11}+U_{12}-D_1L_1+D_2L_2-D_3L_3+D_4L_4,$

$\tilde{\Psi}=[F_3^T\ \ P_{12}\ \ 0\ \ 0\ \ P_{13}\ \ 0\ \ 0\ \ 0\ \ 0\ D_1-D_2\ \ D_3-D_4\ \ 0\ \ 0],$

other $\tilde{\Omega}_{ij}$ are defined as $\tilde{\Omega}_{ij}=\Omega_{ij}$ and other parameters are all defined in Theorem 1.

4. Robust exponential stability results of uncertain delayed neural network

In order to prove the robust stability of the equilibrium point x^* of system (1), we will first simplify system (1) as follows. Let $u(t)=x(t)-x^*$, then we have

$$\dot{u}(t)=-\bar{C}u(t)+\bar{A}f(u(t))+\bar{B}g(u(t-\tau(t)))+\bar{E}\dot{x}(t-\sigma(t)), \quad (9)$$

where $u(t)=(u_1(t),...,u_n(t))^T$, $f_j(u_j(t))=\tilde{f}_j(u_j(t)+x_j^*)-\tilde{f}_j(x_j^*)$, $g_j(u_j(t))=\tilde{g}_j(u_j(t)+x_j^*)-\tilde{g}_j(x_j^*)$ with $f_j(0)=g_j(0)=0$, $j=1,2,...,n.$ By inequalities (2) and (3), we can see that

$$l_{1j}\le\frac{f_j(s)}{s}\le l_{2j},\ l_{3j}\le\frac{g_j(s)}{s}\le l_{4j}. \quad (10)$$

Clearly, the equilibrium point of system (1) is robust stable if and only if the zero solution of system (9) is robust stable.

Now, we present a delay-dependent criterion for the stability of system (1) with time delays $\tau_i(t)=\tilde{\tau}(t)$ $(i=1,...,n)$, where $0\le\tilde{\tau}(t)\le\bar{\tau}.$

Theorem 3. The unique equilibrium point of neural network (1) is robust exponentially stable if the conditions of Theorem 1 are satisfied.

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(t,u_t)=\sum_{i=1}^{7}V_i(t,u_t), \quad (11)$$

With

$$V_1(t,u_t)=e^{2rt}\alpha^T(t)P\alpha(t)+\int_{t-\tilde{\tau}(t)}^{t}e^{2rs}\gamma^T(s)R\gamma(s)\mathrm{d}s,$$

$$V_2(t,u_t)=\int_{t-h_0-h}^{t-h_0}e^{2rs}\beta^T(s)Q\beta(s)\mathrm{d}s+\int_{t-h_0-2h}^{t}e^{2rs}\delta^T(s)U\delta(s)\mathrm{d}s,$$

$$V_3(t,u_t)=h\int_{t-h_0-h}^{t-h_0}\int_{\theta}^{t}e^{2rs}\dot{u}^T(s)S_1\dot{u}(s)\mathrm{d}s\mathrm{d}\theta+h\int_{t-h_0-2h}^{t-h_0-h}\int_{\theta}^{t}e^{2rs}\dot{u}^T(s)S_2\dot{u}(s)\mathrm{d}s\mathrm{d}\theta,$$

$$V_4(t,u_t)=\int_{t-h_0}^{t}e^{2rs}u^T(s)S_3u(s)\mathrm{d}s+h_0\int_{t-h_0}^{t}\int_{\theta}^{t}e^{2rs}\dot{u}^T(s)S_4\dot{u}(s)\mathrm{d}s\mathrm{d}\theta,$$

$$V_5(t,u_t)=\int_{t-\sigma(t)}^{t}e^{2rs}\dot{u}^T(s)S_5\dot{u}(s)\mathrm{d}s+d\int_{t-d}^{t}\int_{\theta}^{t}e^{2rs}\dot{u}^T(s)S_6\dot{u}(s)\mathrm{d}s\mathrm{d}\theta,$$

$$V_6(t,u_t)=2e^{2rt}\sum_{i=1}^{n}\left\{d_{1i}\int_0^{u_i(t)}\left(f_i(s)-l_{1i}s\right)\mathrm{d}s+d_{2i}\int_0^{u_i(t)}\left(l_{2i}s-f_i(s)\right)\mathrm{d}s\right\},$$

$$V_7(t,u_t)=2e^{2rt}\sum_{i=1}^{n}\left\{d_{3i}\int_0^{u_i(t)}\left(g_i(s)-l_{3i}s\right)\mathrm{d}s+d_{4i}\int_0^{u_i(t)}\left(l_{4i}s-g_i(s)\right)\mathrm{d}s\right\},$$

where $\alpha^T(t)=[u^T(t),u^T(t-\tilde{\tau}(t)),u^T(t-h_0-2h)]$, $\beta^T(s)=[u^T(s),u^T(s-h)]$, $\gamma^T(s)=[u^T(s),g^T(u(s)),\dot{u}^T(s)]$, $\delta^T(s)=[u^T(s),\dot{u}^T(s)]$.

For convenience, we denote $u_\tau=u(t-\tilde{\tau}(t))$. The time derivative of functional (11) along the trajectories of system (9) is obtained as follows:

$$\dot{V}_1(t,u_t)=e^{2rt}\{\ 2r\alpha^T(t)P\alpha(t)+2\alpha^T(t)P\dot{\alpha}(t)+\gamma^T(t)R\gamma(t) -(1-\dot{\tilde{\tau}}(t))e^{-2r\tilde{\tau}(t)}\gamma^T(t-\tilde{\tau}(t))R\gamma(t-\tilde{\tau}(t))\}, \quad (12)$$

$$\dot{V}_2(t,u_t)=e^{2rt}\{\ e^{-2rh_0}\beta^T(t-h_0)Q\beta(t-h_0)-e^{-2r(h_0+h)}\beta^T(t-h_0-h)Q\beta(t-h_0-h) +\delta^T(t)U\delta(t)-e^{-2r(h_0+2h)}\delta^T(t-h_0-2h)U\delta(t-h_0-2h)\}, \quad (13)$$

$$\dot{V}_3(t,u_t)=e^{2rt}\{\ h^2\dot{u}^T(t)(S_1+S_2)\dot{u}(t)-h\int_{t-h_0-h}^{t-h_0}e^{2r(s-t)}\dot{u}^T(s)S_1\dot{u}(s)\mathrm{d}s -h\int_{t-h_0-2h}^{t-h_0-h}e^{2r(s-t)}\dot{u}^T(s)S_2\dot{u}(s)\mathrm{d}s\}, \quad (14)$$

$$\dot{V}_4(t,u_t)=e^{2rt}\{\ u^T(t)S_3u(t)-e^{-2rh_0}u^T(t-h_0)S_3u(t-h_0) +h_0^2\dot{u}^T(t)S_4\dot{u}(t)-h_0\int_{t-h_0}^{t}e^{2r(s-t)}\dot{u}^T(s)S_4\dot{u}(s)\mathrm{d}s\}, \quad (15)$$

$$\dot{V}_5(t,u_t)=e^{2rt}\{\ \dot{u}^T(t)S_5\dot{u}(t)-(1-\dot{\sigma}(t))e^{-2r\sigma(t)}\dot{u}^T(t-\sigma(t))S_5\dot{u}(t-\sigma(t)) +d^2\dot{u}^T(t)S_6\dot{u}(t)-d\int_{t-d}^{t}e^{2r(s-t)}\dot{u}^T(s)S_6\dot{u}(s)\mathrm{d}s\}, \quad (16)$$

$$\dot{V}_6(t,u_t)=e^{2rt}\{\ 4r\sum_{i=1}^{n}\{d_{1i}\int_0^{u_i(t)}\left(f_i(s)-l_{1i}s\right)\mathrm{d}s+d_{2i}\int_0^{u_i(t)}\left(l_{2i}s-f_i(s)\right)\mathrm{d}s\} +2\{f^T(u(t))-u^T(t)L_1\}D_1\dot{u}(t)+2\{u^T(t)L_2-f^T(u(t))\}D_2\dot{u}(t)\}, \quad (17)$$

$$\dot{V}_7(t,u_t)=e^{2rt}\Big\{\ 4r\sum_{i=1}^{n}\Big\{d_{3i}\int_0^{u_i(t)}\big(g_i(s)-l_{3i}s\big)\mathrm{d}s+d_{4i}\int_0^{u_i(t)}\big(l_{4i}s-g_i(s)\big)\mathrm{d}s\Big\}$$
$$+2\{g^T(u(t))-u^T(t)L_3\}D_3\dot{u}(t)+2\{u^T(t)L_4-g^T(u(t))\}D_4\dot{u}(t)\Big\}. \quad (18)$$

From inequalities (10) and Lemma 3, we have

$$\sum_{i=1}^{n} d_{1i}\int_0^{u_i(t)}\big(f_i(s)-l_{1i}s\big)\mathrm{d}s \le \{f^T(u(t))-u^T(t)L_1\}D_1u(t) \le u^T(t)(L_2-L_1)D_1u(t), \quad (19)$$

$$\sum_{i=1}^{n} d_{2i}\int_0^{u_i(t)}\big(l_{2i}s-f_i(s)\big)\mathrm{d}s \le \{u^T(t)L_2-f^T(u(t))\}D_2u(t) \le u^T(t)(L_2-L_1)D_2u(t), \quad (20)$$

$$\sum_{i=1}^{n} d_{3i}\int_0^{u_i(t)}\big(g_i(s)-l_{3i}s\big)\mathrm{d}s \le \{g^T(u(t))-u^T(t)L_3\}D_3u(t) \le u^T(t)(L_4-L_3)D_3u(t), \quad (21)$$

$$\sum_{i=1}^{n} d_{4i}\int_0^{u_i(t)}\big(l_{4i}s-g_i(s)\big)\mathrm{d}s \le \{u^T(t)L_4-g^T(u(t))\}D_4u(t) \le u^T(t)(L_4-L_3)D_4u(t). \quad (22)$$

On the other hand, one can infer from inequalities (10) that the following matrix inequalities hold for any positive diagonal matrices $T_i(i=1,2,3)$ with compatible dimensions

$$0\le -2e^{2rt}\Big\{\ u^T(t)L_2T_1L_1u(t)+f^T(u(t))T_1f(u(t))-u^T(t)T_1(L_1+L_2)f(u(t))\Big\}, \quad (23)$$
$$0\le -2e^{2rt}\Big\{\ u^T(t)L_4T_2L_3u(t)+g^T(u(t))T_2g(u(t))-u^T(t)T_2(L_3+L_4)g(u(t))\Big\}, \quad (24)$$
$$0\le -2e^{2rt}\Big\{\ u_\tau^TL_4T_3L_3u_\tau+g^T(u_\tau)T_3g(u_\tau)-u_\tau^TT_3(L_3+L_4)g(u_\tau)\Big\}. \quad (25)$$

Moreover, by using the Jensen integral inequality (Lemma 4), we obtain

$$-h_0\int_{t-h_0}^{t}\dot{u}^T(s)S_4\dot{u}(s)\mathrm{d}s \le -\Big(\int_{t-h_0}^{t}\dot{u}(s)\mathrm{d}s\Big)^TS_4\int_{t-h_0}^{t}\dot{u}(s)\mathrm{d}s = -\big(u^T(t)-u^T(t-h_0)\big)S_4\big(u(t)-u(t-h_0)\big), \quad (26)$$

$$-d\int_{t-d}^{t}\dot{u}^T(s)S_6\dot{u}(s)\mathrm{d}s \le -\Big(\int_{t-d}^{t}\dot{u}(s)\mathrm{d}s\Big)^TS_6\int_{t-d}^{t}\dot{u}(s)\mathrm{d}s = -\big(u^T(t)-u^T(t-d)\big)S_6\big(u(t)-u(t-d)\big). \quad (27)$$

To get less conservative criterion, we introduce the following equality for any real matrix X_0 with compatible dimension

$$0=2\dot{u}^T(t)X_0^T\big\{-\dot{u}(t)-\bar{C}u(t)+\bar{A}f(u(t))+\bar{B}g(u_\tau)+\bar{E}\dot{u}(t-\sigma(t))\big\}. \quad (28)$$

Next, we will discuss the variation of derivatives of $V(t,u_t)$ for two cases, i.e., $h_0\le\tilde{\tau}(t)\le h_0+h$ and $h_0+h\le\tilde{\tau}(t)\le h_0+2h$, respectively.

Case I: $h_0\le\tilde{\tau}(t)\le h_0+h$.

Again from Lemma 4 we have the following matrix inequalities:

$$-h\int_{t-h_0-2h}^{t-h_0-h}\dot{u}^T(s)S_2\dot{u}(s)\mathrm{d}s$$
$$\le -\Big(\int_{t-h_0-2h}^{t-h_0-h}\dot{u}(s)\mathrm{d}s\Big)^TS_2\int_{t-h_0-2h}^{t-h_0-h}\dot{u}(s)\mathrm{d}s$$
$$= -\big(u^T(t-h_0-h)-u^T(t-h_0-2h)\big)S_2\big(u(t-h_0-h)-u(t-h_0-2h)\big). \quad (29)$$

Based on Leibniz-Newton formula, for any real matrix $X_i(i=1,\dots,4)$ with compatible dimensions, we get

$$0=\ 2e^{2rt}\big\{u_\tau^TX_1^T+u^T(t-h_0)X_2^T\big\}\Big\{u(t-h_0)-u_\tau-\int_{t-\tilde{\tau}(t)}^{t-h_0}\dot{u}(s)\mathrm{d}s\Big\}, \quad (30)$$

$$0=\ 2e^{2rt}\big\{u_\tau^TX_3^T+u^T(t-h_0-h)X_4^T\big\}\Big\{u_\tau-u(t-h_0-h)-\int_{t-h_0-h}^{t-\tilde{\tau}(t)}\dot{u}(s)\mathrm{d}s\Big\}. \quad (31)$$

It is easy to get the following inequalities by using Lemma 1:

$$-2\big\{u_\tau^TX_1^T+u^T(t-h_0)X_2^T\big\}\int_{t-\tilde{\tau}(t)}^{t-h_0}\dot{u}(s)\mathrm{d}s$$
$$\le\ he^{-2r(h_0+h)}\int_{t-\tilde{\tau}(t)}^{t-h_0}\dot{u}^T(s)S_1\dot{u}(s)\mathrm{d}s$$
$$+\frac{1}{h}e^{2r(h_0+h)}(\tilde{\tau}(t)-h_0)\zeta^T(t)\Gamma_{11}^TS_1^{-1}\Gamma_{11}\zeta(t), \quad (32)$$

$$-2\big\{u_\tau^TX_3^T+u^T(t-h_0-h)X_4^T\big\}\int_{t-h_0-h}^{t-\tilde{\tau}(t)}\dot{u}(s)\mathrm{d}s$$
$$\le\ he^{-2r(h_0+h)}\int_{t-h_0-h}^{t-\tilde{\tau}(t)}\dot{u}^T(s)S_1\dot{u}(s)\mathrm{d}s$$
$$+\frac{1}{h}e^{2r(h_0+h)}(h_0+h-\tilde{\tau}(t))\zeta^T(t)\Gamma_{12}^TS_1^{-1}\Gamma_{12}\zeta(t), \quad (33)$$

where

$$\zeta^T(t)=[u^T(t),u_\tau^T,u^T(t-h_0),u^T(t-h_0-h),u^T(t-h_0-2h),(1-\dot{\tilde{\tau}}(t))\dot{u}_\tau^T,$$
$$\dot{u}^T(t-h_0-2h),u^T(t-d),\dot{u}^T(t-\sigma(t)),f^T(u(t)),g^T(u(t)),g^T(u_\tau),\dot{u}^T(t)].$$

From (12)-(33) and by applying S-procedure [1] we obtain Based on Lemma 5, from conditions (4) and (5) the following matrix inequality holds

$$\Omega\ +\Sigma_1+\Psi^TH_0E_0(t)\Theta+\Theta^TE_0^T(t)H_0^T\Psi$$
$$+\frac{1}{h}e^{2r(h_0+h)}(\tilde{\tau}(t)-h_0)\Gamma_{11}^TS_1^{-1}\Gamma_{11}+\frac{1}{h}e^{2r(h_0+h)}(h_0+h-\tilde{\tau}(t))\Gamma_{12}^TS_1^{-1}\Gamma_{12}<0$$

if and only if the following matrix inequality holds for any positive scalar ε

$$\Omega\ +\Sigma_1+\varepsilon\Theta^T\Theta+\varepsilon^{-1}\Psi^TH_0H_0^T\Psi$$
$$+\frac{1}{h}e^{2r(h_0+h)}(\tilde{\tau}(t)-h_0)\Gamma_{11}^TS_1^{-1}\Gamma_{11}+\frac{1}{h}e^{2r(h_0+h)}(h_0+h-\tilde{\tau}(t))\Gamma_{12}^TS_1^{-1}\Gamma_{12}<0.$$

Note that $h_0\le\tilde{\tau}(t)\le h_0+h,$ so above matrix inequality holds if and only if

$$\Omega+\Sigma_1+\varepsilon\Theta^T\Theta+\varepsilon^{-1}\Psi^TH_0H_0^T\Psi+e^{2r(h_0+h)}\Gamma_{11}^TS_1^{-1}\Gamma_{11}<0 \quad (34)$$

and

$$\Omega+\Sigma_1+\varepsilon\Theta^T\Theta+\varepsilon^{-1}\Psi^TH_0H_0^T\Psi+e^{2r(h_0+h)}\Gamma_{12}^TS_1^{-1}\Gamma_{12}<0 \quad (35)$$

are true. From the well known Schur complement, inequalities (34),(35) are equivalent to (6) with i=1,j=1 and
j=2 respectively, thus $\dot{V}(t,u_t)<0$ holds if (6) (i=1,j=1,2) are true.

Case II: $h_0+h\le\tilde{\tau}(t)\le h_0+2h$.

Again from Lemma 4 we have the following matrix inequalities:

$$-h\int_{t-h_0-h}^{t-h_0} \dot{u}^T(s)S_1\dot{u}(s)\mathrm{d}s$$
$$\le -\Big(\int_{t-h_0-h}^{t-h_0} \dot{u}(s)\mathrm{d}s\Big)^T S_1\int_{t-h_0-h}^{t-h_0} \dot{u}(s)\mathrm{d}s$$
$$= -\big(u^T(t-h_0)-u^T(t-h_0-h)\big)S_1\big(u(t-h_0)-u(t-h_0-h)\big). \quad (36)$$

According to Leibniz-Newton formula, for any real matrix $Y_i(i=1,...,4)$ with compatible dimensions, we get

$$0= 2e^{2rt}\{u_\tau^TY_1^T+u^T(t-h_0-h)Y_2^T\}\{u(t-h_0-h)-u_\tau-\int_{t-\tilde{\tau}(t)}^{t-h_0-h}\dot{u}(s)\mathrm{d}s\}, \quad (37)$$

$$0= 2e^{2rt}\{u_\tau^TY_3^T+u^T(t-h_0-2h)Y_4^T\}\{u_\tau-u(t-h_0-2h)-\int_{t-h_0-2h}^{t-\tilde{\tau}(t)}\dot{u}(s)\mathrm{d}s\}. \quad (38)$$

From Lemma 1, the following inequalities hold:

$$-2\{u_\tau^TY_1^T+u^T(t-h_0-h)Y_2^T\}\int_{t-\tilde{\tau}(t)}^{t-h_0-h}\dot{u}(s)\mathrm{d}s$$
$$\le \quad he^{-2r(h_0+2h)}\int_{t-\tilde{\tau}(t)}^{t-h_0-h}\dot{u}^T(s)S_2\dot{u}(s)\mathrm{d}s$$
$$+\frac{1}{h}e^{2r(h_0+2h)}(\tilde{\tau}(t)-h-h_0)\zeta^T(t)\Gamma_{21}^TS_2^{-1}\Gamma_{21}\zeta(t), \quad (39)$$
$$-2\{u_\tau^TY_3^T+u^T(t-h_0-2h)Y_4^T\}\int_{t-h_0-h}^{t-\tilde{\tau}(t)}\dot{u}(s)\mathrm{d}s$$
$$\le \quad he^{-2r(h_0+2h)}\int_{t-h_0-2h}^{t-\tilde{\tau}(t)}\dot{u}^T(s)S_2\dot{u}(s)\mathrm{d}s$$
$$+\frac{1}{h}e^{2r(h_0+2h)}(h_0+2h-\tilde{\tau}(t))\zeta^T(t)\Gamma_{22}^TS_2^{-1}\Gamma_{22}\zeta(t), \quad (40)$$

From (12)-(28) and (36)-(40) and by applying S-procedure [1] we obtain

$$\dot{V}(t,u_t)\le e^{2rt}\zeta^T(t)\Big(\Omega +\Sigma_2+\Psi^TH_0E_0(t)\Theta+\Theta^TE_0^T(t)H_0^T\Psi+\frac{1}{h}e^{2r(h_0+2h)}$$
$$\times(\tilde{\tau}(t)-h_0-h)\Gamma_{21}^TS_2^{-1}\Gamma_{21}+\frac{1}{h}e^{2r(h_0+2h)}(h_0+2h-\tilde{\tau}(t))\Gamma_{22}^TS_2^{-1}\Gamma_{22}\Big)\zeta(t).$$

Based on Lemma 5, from conditions (4) and (5) the following matrix inequality holds

$$\Omega +\Sigma_2+\Psi^TH_0E_0(t)\Theta+\Theta^TE_0^T(t)H_0^T\Psi$$
$$+\frac{1}{h}e^{2r(h_0+2h)}(\tilde{\tau}(t)-h_0-h)\Gamma_{21}^TS_2^{-1}\Gamma_{21}+\frac{1}{h}e^{2r(h_0+2h)}(h_0+2h-\tilde{\tau}(t))\Gamma_{22}^TS_2^{-1}\Gamma_{22}<0$$

if and only if the following matrix inequality holds for any positive scalar ε

$$\Omega +\Sigma_2+\varepsilon\Theta^T\Theta+\varepsilon^{-1}\Psi^TH_0H_0^T\Psi$$
$$+\frac{1}{h}e^{2r(h_0+2h)}(\tilde{\tau}(t)-h_0-h)\Gamma_{21}^TS_2^{-1}\Gamma_{21}+\frac{1}{h}e^{2r(h_0+2h)}(h_0+2h-\tilde{\tau}(t))\Gamma_{22}^TS_2^{-1}\Gamma_{22}<0.$$

Note that $h_0+h\le\tilde{\tau}(t)\le h_0+2h$, so above matrix inequality holds if and only if

$$\Omega+\Sigma_2+\varepsilon\Theta^T\Theta+\varepsilon^{-1}\Psi^TH_0H_0^T\Psi+e^{2r(h_0+2h)}\Gamma_{21}^TS_2^{-1}\Gamma_{21}<0 \quad (41)$$

and

$$\Omega+\Sigma_2+\varepsilon\Theta^T\Theta+\varepsilon^{-1}\Psi^TH_0H_0^T\Psi+e^{2r(h_0+2h)}\Gamma_{22}^TS_2^{-1}\Gamma_{22}<0 \quad (42)$$

are true. From the well known Schur complement, inequalities (41),(42) are equivalent to (6) with i=2,j=1 and j=2 respectively, thus $\dot{V}(t,u_t)<0$ holds if (6) (i=2,j=1,2) are true.

Furthermore, following the similar line in [6], from Lemma 1 we have

$$V(t,u_t)|_{t=0}\le M_1\|\phi(t)-x^*\|^2+M_2\sup_{-\bar{h}\le\theta\le0}\|\dot{u}(\theta)\|^2,$$

where

$$M_1 = 9\lambda_M(P)+4h\lambda_M(Q)+3\bar{h}\lambda_M(R)\big(1+\sigma_M^2\big)+2\bar{h}\lambda_M(U)$$
$$+2\lambda_M(D_1+D_2)\lambda_M(L_2-L_1)+2\lambda_M(D_3+D_4)\lambda_M(L_4-L_3)+h_0\lambda_M(S_3),$$
$$M_2 = 3\bar{h}\lambda_M(R)+2\bar{h}\lambda_M(U)+h^T\bar{h}\big(\lambda_M(S_1)+\lambda_M(S_2)\big)$$
$$+\frac{1}{2}h_0^2\lambda_M(S_4)+d\lambda_M(S_5)+\frac{1}{2}d^2\lambda_M(S_6),$$

and $\sigma_M=\max_{1\le i\le n}\{|l_{3i}|,|l_{4i}|\}$.

Meanwhile $V(t,u_t)\ge e^{2rt}\|\phi(t)-x^*\|^2\lambda_m(P_{11})$, by Lyapunov stability theory, the proof of Theorem 1 is completed.

Remark 1. In Theorems 1 and 3, by setting $P_{i2}=P_{2i}=0(i=1,2,3), R=0$, we can employ this criterion to analyze the existence, uniqueness and stability of neural network (1) when $\dot{\tilde{\tau}}(t)\ge1$ or $\dot{\tilde{\tau}}(t)$ is unknown.

Remark 2. It is easy to see that the derivatives of $\alpha^T(t)P\alpha(t)$ and $\int_{t-\tilde{\tau}(t)}^{t}\gamma^T(s)R\gamma(s)\mathrm{d}s$ have some terms containing $1-\dot{\tilde{\tau}}(t)$. In order to absorb some $1-\dot{\tilde{\tau}}(t)$, we introduce $(1-\dot{\tilde{\tau}}(t))\dot{u}^T(t-\tilde{\tau}(t))$ in $\zeta(t)$ but not $\dot{u}^T(t-\tilde{\tau}(t))$, so Ω contains fewer $1-\dot{\tilde{\tau}}(t)$, which leads to a more effective result [14].

Remark 3. If h_0 is zero, by choosing the Lyapunov functional candidate as defined in (11) with $S_3=S_4=0, h_0=0$, using the similar method shown in the proof of Theorems 1,3 and Remark 1, we can obtain a criterion to verify the existence, uniqueness and global stability of system (1).

Remark 4. If $\tilde{f}(s)=\tilde{g}(s)$, by choosing functional (11) with $D_1=D_2=T_1=0$, similar to Theorems 1,3 and Remark 1, we can obtain a criterion to verify the existence, uniqueness and global stability of system (1).

Remark 5. If $f_j(s)=g_j(s), h_0=0$, by choosing functional (11) with $D_1=D_2=T_1=S_3=S_4=0, h_0=0$, similar to Theorems 1,3 and Remark 1, we can obtain a criterion to verify the existence, uniqueness and global stability of system (1).

Next, we consider the stability of system (1) with different time delays $\tau_i(t)(i=1,...,n)$ satisfying $0\le h_0\le\tau_i(t)\le h_0+2h$.

Theorem 4. The unique equilibrium point of neural network (1) is robust exponentially stable if the conditions of Theorem 2 are satisfied.

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(t,u_t)=\sum_{i=2}^{8}V_i(t,u_t)$$

with

$$V_8(t,u_t)= e^{2rt}\xi^T(t)P\xi(t)+\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t}\left(r_{1i}u_i^2(s)+r_{2i}f_i^2(u_i(s))+r_{3i}\dot{u}_i^2(s)\right)ds,$$

where $\xi^T(t)=[u^T(t),u^T(t-\tau(t)),u^T(t-h_0-2h)]$. The time derivatives of $V_8(t,u_t)$ along the trajectories of system (9) satisfy:

$$\dot{V}_8(t,u_t)=e^{2rt}\{2r\xi^T(t)P\xi(t)+2\xi^T(t)P\dot{\xi}(t)+u^T(t)R_1u(t)+f^T(u(t))R_2f^T(u(t))$$
$$+\dot{u}^T(t)R_3\dot{u}(t)-\sum_{i=1}^{n}(1-\dot{\tau}_i(t))\left(r_{1i}u_i^2(t-\tau_i(t))+r_{2i}f_i^2(u(t-\tau_i(t)))+r_{3i}\dot{u}_i^2(t-\tau_i(t))\right)\}.$$

Set

$$\begin{aligned}\zeta^T(t)=[\quad & u^T(t),u^T(t-\tau(t)),u^T(t-h_0),u^T(t-h_0-h),\\ & u^T(t-h_0-2h),\dot{u}^T(t-\vec{\tau}(t)),\dot{u}^T(t-h_0-2h),u^T(t-d),\\ & \dot{u}^T(t-\sigma(t)),\ f^T(u(t)),g^T(u(t)),g^T(u(t-\tau(t))),\dot{u}^T(t)\ ],\end{aligned}$$

$$\dot{u}^T(t-\vec{\tau}(t))=[\ (1-\dot{\tau}_1(t))\dot{u}_1^T(t-\tau_1(t)),\dots,(1-\dot{\tau}_n(t))\dot{u}_n^T(t-\tau_n(t))\ ],$$

we can complete this proof in the similar way as the proof of Theorem 3.

Remark 6. In Theorems 2 and 4, if we set $P_{i2}=P_{2i}=0$, $R_i=0(i=1,2,3)$, by deleting $\dot{u}^T(t-\vec{\tau}(t))$ from $\zeta^T(t)$, we can employ this criterion to analyze the existence, uniqueness and stability of neural network (1) when some $\dot{\tau}_i(t)\geq 1$ or some $\dot{\tau}_i(t)$ is unknown, i=1,...,n.

Remark 7. If $h_0=0$ or $\tilde{f}(x)=\tilde{g}(x)$ in neural network (1) with different time delays $\tau_i(t)(i=1,\dots,n)$, similar to Remarks 4-6, we can derive criteria to analyze the existence, uniqueness and stability of neural networks (1).

5. Comparison and Illustrative Examples

Now, we provide three numerical examples to demonstrate the effectiveness and less conservativeness of our delay-dependent stability criteria over some recent results in the literature.

Example 1. Consider system (9) with $\tau_i(t)=\tilde{\tau}(t)$ $(i=1,\dots,4)$ and

$C=\mathrm{diag}\{1.2769,0.6231,0.9230,0.4480\}$,

$$A=\begin{bmatrix}-0.0373 & 0.4852 & -0.3351 & 0.2336\\ -1.6033 & 0.5988 & -0.3224 & 1.2352\\ 0.3394 & -0.0860 & -0.3824 & -0.5785\\ -0.1311 & 0.3253 & -0.9534 & -0.5015\end{bmatrix},$$

$$B=\begin{bmatrix}0.8674 & -1.2405 & -0.5325 & 0.0220\\ 0.0474 & -0.9164 & 0.0360 & 0.9816\\ 1.8495 & 2.6117 & -0.3788 & 0.8428\\ -2.0413 & 0.5179 & 1.1734 & -0.2775\end{bmatrix},$$

$\Delta C(t)=\Delta A(t)=\Delta B(t)=\bar{E}=0,$

$\tilde{f}(x)=\tilde{g}(x), L_1=0, L_2=\mathrm{diag}\{0.1137,0.1279,0.7994,0.2368\}.$

This model was studied in [5,7,23]. Obviously,

$$D=C-(\tilde{A}+|B|)\Sigma=\begin{bmatrix}1.1825 & -0.2207 & -0.6936 & -0.0605\\ -0.1877 & 0.4293 & -0.2865 & -0.5249\\ -0.2489 & -0.3450 & 0.9259 & -0.3366\\ -0.2470 & -0.1078 & -1.7002 & 0.5010\end{bmatrix}$$

is not an M-matrix, where $\tilde{A}=[\tilde{a}_{ij}]_{n\times n}, |B|=[|b_{ij}|]_{n\times n}$, and

$$\tilde{a}_{ij}=\begin{cases}a_{ii} & i=j,\\ |a_{ij}| & i\neq j.\end{cases}$$

Therefore the stability of this model can't be ascertained by using Theorem 1 in [12].

Further, it is verified that all of the conditions given in [13,15,17,19] admit no feasible solutions for any positive time delay or given η. That is, none of the criteria given in [13,15,17,19] can conclude whether this model is stable or not.

Moreover, if we set exponential convergence rate r be fixed as 0.24, all of the results given in [6,16] fail to ascertain the stability of this system. However, by Remark 5, we can obtain that, for constant time delay and r=0.24, the origin of this system is the unique equilibrium point which is exponential stable for any time delay with $\tilde{\tau}(t)\leq 0.7845$.

In addition, if we set time delay $\tilde{\tau}(t)$ be fixed as 1, the maximal exponential convergence rates of k in [6, 16] are all 0.1705. However, from Remark 5 we can confirm that the equilibrium point of this system is unique and exponential stable with convergence rate of r=0.2237. Furthermore, it is assumed that the exponential convergence rate k is fixed as zero (i.e. asymptotical stability), the maximal upper bounds of time delay $\tilde{\tau}(t)$ for various η's from Remarks 5 and 6 in this paper and those in [5–7,16,23] are listed in Table I, where ``unknown η '' means that η can be arbitrary value or $\tilde{\tau}(t)$ can be not differentiable. It is clear that the results in this paper are markedly better than those in [5–7,16,23].

Table I Calculated maximal upper bounds of time delays $\tilde{\tau}(t)$ for various η of Example 1 with r=0

h_0	methods	$\eta=0$	$\eta=0.5$	$\eta=0.9$	unknown η
0	[5,16]	3.5841	2.5376	2.0853	2.0389
0	[23]	3.7327	2.5943	2.1306	2.0770
0	[7]	4.0945	2.7353	2.2760	2.1326
0	Remark 5	4.5838	4.0245	3.2947	2.3746
1	[5]	3.5841	2.5802	2.2736	2.2393
1	Remark 4	4.6545	4.0519	3.4141	2.4978
2	[5]	3.5841	2.7500	2.6468	2.6298
2	Remark 4	4.6620	4.0519	3.5042	2.7780

Example 2. Consider system (1) with constant time delay $\tau_i(t)=\sigma(t)=d\,(i=1,2,3)$ and

$$C=\mathrm{diag}\{2.7644,1.0185,10.2716\},$$

$$A=\begin{bmatrix}0.2651 & -3.1608 & -2.0491\\ 3.1859 & -0.1573 & -2.4687\\ 2.0368 & -1.3633 & 0.5776\end{bmatrix},$$

$$B=\begin{bmatrix}-0.7727 & -0.8370 & 3.8019\\ 0.1004 & 0.6677 & -2.4431\\ -0.6622 & 1.3109 & -1.8407\end{bmatrix},$$

$$E=\begin{bmatrix}0.2076 & 0.0631 & 0.3915\\ -0.0780 & 0.3106 & 0.1009\\ -0.2763 & 0.1416 & 0.3729\end{bmatrix},$$

$$\Delta C(t)=\Delta A(t)=\Delta B(t)=\Delta E(t)=0,$$

$$\tilde{f}(x)=\tilde{g}(x), L_1=0,\ L_2=\mathrm{diag}\{0.1019,0.3419,0.0633\}.$$

This model was studied in [9, 18]. Ref. [18] illustrated that the maximum bound of delays is 1.0344. Let m=3 in [9], the authors obtained the upper bound of delay is 82. However by using our Remark 5 to this example, we can obtain the system is feasible for any d>0. It means that the system is delay-independent stable, which shows that our criteria are less conservative than [9,18].

Example 3. Consider system (9) with

$$C=1.5I,\ A=\begin{bmatrix}0.5 & 0.75\\ 0 & -0.5\end{bmatrix}, B=\begin{bmatrix}0.4 & 0.5\\ 0 & -0.5\end{bmatrix}, E=\begin{bmatrix}0.4 & -0.1\\ 0.2 & 0.45\end{bmatrix},$$

$$H_0=0.5I,\ \ E_0(t)=\begin{bmatrix}\cos(2t) & 0\\ 0 & \cos(3t)\end{bmatrix},\ G_0=\begin{bmatrix}-0.2 & 0.5\\ 0.3 & 0.5\end{bmatrix},$$

$$G_1=\begin{bmatrix}0.3 & -0.5\\ 0.5 & 0\end{bmatrix},\ G_2=\begin{bmatrix}0.3 & -0.5\\ 0.5 & 0.3\end{bmatrix},\ \ G_3=\begin{bmatrix}0.5 & 0\\ -0.5 & 0.5\end{bmatrix},$$

$$\tilde{f}_1(s)=\tilde{f}_2(s)=\frac{1}{2}(|s+1|-|s-1|),\tilde{g}_1(s)=\tilde{g}_2(s)=\frac{1}{4}(|s+1|-|s-1|).$$

Thus the neural activation functions satisfy the inequalities (2) and (3) with $L_1=L_3=0, L_2=I, L_4=0.5I$. For the case with $\tau_i(t)=\tilde{\tau}(t)(i=1,2)$, if we set $r=0,\eta=0.9$, $\sigma(t)=1$, from Remark 3 we can obtain that the system has a unique equilibrium point which is robust stable for any time delay with $0\le\tilde{\tau}(t)<2.6653$.

For the case with different time delays $\tau_1(t)\neq\tau_2(t)$, if we set $r=0,\sigma(t)=1$, from Remark 7 we can obtain that the system has a unique equilibrium point which is robust stable for any time delay with $0\le\tau_i(t)<2.0768$ even if any $\dot{\tau}_i(t)\ge 1$ or any $\dot{\tau}_i(t)(i=1,2)$ are unknown.

Therefore, we can say that for these three systems the results in this paper are much effective and less conservative than those in [5–7,9,12,13,15–19,23].

6. Conclusions

In this paper we have investigated the uniqueness and global robust stability problem of uncertain neural networks of neutral-type. By employing new Lyapunov Krasovskii functional, we proposed several novel stability criteria for the considered systems. The obtained results are all in the form of LMIs, which can be easily optimized. Finally, three examples are given to show the superiority of our proposed stability conditions to some existing ones.

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