Robust stability condition for neutral-type neural networks with discrete and distributed delays

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Abstract

The robust exponential stability is investigated for a class of uncertain neutral-type neural networks with discrete and distributed time-varying delays. By introducing a new vector Lyapunov-Krasovskii functional, using Jensen integral inequality, free-weighting matrix method and linear matrix inequality techniques, delay-dependent sufficient conditions are obtained for exponential stability of considered neural networks, which generalize some previous results in the literature. Four examples are given to show the less conservativeness of the obtained results.

Keywords: *Global robust exponential stability; linear matrix inequality(LMI); uncertain neutral-type neural networks; Jensen integral inequality; free-weighting matrix.*

1. Introduction

In recent decades, neural networks have been successfully applied to various fields such as optimization, image processing and associative memory design. In such application, it is important to know the stability properties of the designed neural network, these properties include asymptotic stability and exponential stability. However, time delays inevitably exist in neural networks due to various reasons [13]. The existence of time delay may lead to some complex dynamic behaviors such as oscillation, divergence, chaos, instability or other poor performance of the neural networks. Therefore, stability analysis for neural networks with delays has attracted more and more interests in recent years. Various sufficient conditions, either delay-dependent or delay-independent, have been proposed to guarantee the global stability for neural networks with constant and time-varying delays, for example, see [9,12, 17–19,24] and references therein.

Since neural networks usually have a spatial extent, there is a distribution of propagation delays over a period of time. In these circumstances the signal propagation is not instantaneous and can't be modeled with discrete delays [6]. A more appropriate way is to incorporate continuously distributed delays in neural network model [2,5,7,10, 13,16]. On the other hand, uncertainties are inevitable in neural networks because of the existence of modeling errors, external disturbance and parameter fluctuation in the process of implementations. Therefore it is important to study the robust stability of delayed neural networks in the presence of uncertainties [8,14,20–23].

Motivated by the above discussions, in this paper we consider a class of uncertain neutral-type neural networks with discrete and distributed time-varying delays. Based on a new vector Lyapunov-Krasovskii functional, delay-dependent sufficient conditions are obtained for exponential stability of considered neural networks. By using Jensen integral inequality, free-weighting matrix method and LMI techniques, the results are less conservative than some previous ones in the literature. Four examples are given to show the effectiveness of the obtained results.

2. Problem description

Considering the following uncertain neutral-type neural networks with discrete and distributed time-varying delays:

$$\begin{aligned}\dot z(t) = & -\bar C z(t) + \bar A \tilde f(z(t)) + \bar B \tilde f(z(t-\tau(t))) \\ & + \bar D \int_{t-\sigma(t)}^{t} \tilde g(z(s))\mathrm{d}s + \bar E \dot z(t-\varpi) + J,\end{aligned} \tag{1}$$

where $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T \in \mathrm{R}^n$ is the neural state vector, $\bar C = C + \Delta C(t)$, $\bar A = A + \Delta A(t)$, $\bar B = B + \Delta B(t)$, $\bar D = D + \Delta D(t)$, $\bar E = E + \Delta E(t)$. $C = \mathrm{diag}\{c_1, c_2, \dots, c_n\}$ is a positive diagonal matrix, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $D = (d_{ij})_{n\times n}$, $E = (e_{ij})_{n\times n}$ are known constant matrices, $\Delta C(t)$, $\Delta A(t), \Delta B(t), \Delta D(t), \Delta E(t)$ are parametric uncertainties, $0 \le \tau(t) \le \bar\tau$, $0 \le \sigma(t) \le \bar\sigma, \varpi$ are time-varying delays, where $\bar\tau, \bar\sigma, \varpi$ are constants. J is the constant external input vector, and $\tilde f(z(t)) = (\tilde f_1(z_1(t)), \tilde f_2(z_2(t)), \dots, \tilde f_n(z_n(t)))^T$, $\tilde g(z(t)) = (\tilde g_1(z_1(t)), \tilde g_2(z_2(t)), \dots, \tilde g_n(z_n(t)))^T \in \mathrm{R}^n$ denote the neural activation functions. It is assumed that $\tilde f_j(z_j(t))$, $\tilde g_j(z_j(t))$ are bounded and there exist constants l_{1j}, l_{2j} such that

$$\begin{aligned}|\tilde f_j(x) - \tilde f_j(y)| \le l_{1j}|x-y|, \\ |\tilde g_j(x) - \tilde g_j(y)| \le l_{2j}|x-y|\end{aligned} \tag{2}$$

for any $x, y \in \mathrm{R}$, $x \ne y$, $j = 1, 2, \dots, n$.

Moreover, we assume that the initial condition of system (1) has the form

$$z_i(t) = \phi_i(t), \quad t \in [-\max\{\bar{\tau}, \bar{\sigma}, \varpi\}, 0]$$

where $\phi_i(t)(i = 1, 2, ..., n)$ are continuous functions.

From the well-known Brouwer's fixed point theorem, system (1) always has an equilibrium point z^* [13].
Throughout this paper, let ||y|| denote the Euclidean norm of a vector $y \in R^n, W^T, W^{-1}, \lambda_M(W), \lambda_m(W)$ and $\|W\| = \sqrt{\lambda_M(W^T W)}$ denote the transpose, the inverse, the largest eigenvalue, the smallest eigenvalue, and the spectral norm of a square matrix W, respectively. Let W>0(<0) denote a positive (negative) definite symmetric matrix, I denote an identity matrix with compatible dimension.

In order to prove the robust stability of the equilibrium point z^* of system (1), we will first simplify system (1) as follows. Let $u(\cdot) = z(\cdot) - z^*$, then we have

$$\begin{aligned} \dot{u}(t) = & -\bar{C}u(t) + \bar{A}f(u(t)) + \bar{B}f(u(t-\tau(t))) \\ & + \bar{D}\int_{t-\sigma(t)}^{t} g(u(s))\mathrm{d}s + \bar{E}\dot{u}(t-\varpi), \end{aligned} \tag{3}$$

where $u(t) = (u_1(t), u_2(t), ..., u_n(t))^T$, $f_j(u_j(t)) = \tilde{f}_j(u_j(t) + z_j^*) - \tilde{f}_j(z_j^*), g_j(u_j(t)) = \tilde{g}_j(u_j(t) + z_j^*) - \tilde{g}_j(z_j^*)$ with $f_j(0) = 0, \quad g_j(0) = 0, \ j = 1, 2, ..., n.$ By assumption (2), we can see that

$$\begin{aligned} |f_j(x) - f_j(y)| &\le l_{1j}|x - y|, \\ |g_j(x) - g_j(y)| &\le l_{2j}|x - y|. \end{aligned} \tag{4}$$

Suppose that the time-varying uncertain matrices $\Delta C(t)$, $\Delta A(t), \Delta B(t), \Delta D(t), \Delta E(t)$ are norm-bounded, which are in the form of

$$\begin{aligned} &[\Delta C(t) \quad \Delta A(t) \quad \Delta B(t) \quad \Delta D(t) \quad \Delta E(t)] \\ &= H_0 F_0(t)[G_0 \quad G_1 \quad G_2 \quad G_3 \quad G_4], \end{aligned} \tag{5}$$

where $H_0, G_0, G_1, G_2, G_3, G_4$ are known constant real matrices with appropriate dimensions, the uncertainty $F_0(t)$ is defined as

$$F_0^T(t)F_0(t) \le I. \tag{6}$$

The definition of exponential stability is now given.
Definition 1: The system (1) is said to be globally exponentially stable if there exist constants k>0 and M>1 such that

$$\|x(t)\| \le M \sup_{-\max\{\bar{\tau},\bar{\sigma},\varpi\} \le \theta \le 0} \left(\|x(\theta)\|, \|\dot{x}(\theta)\|\right) e^{-kt},$$

where k is called the exponential convergence rate.
Clearly, the equilibrium point of system (1) is robust stable if and only if the zero solution of system (3) is robust stable.

In order to obtain the results, we need the following lemmas.
Lemma 1 (see [3]) For any positive symmetric constant matrix $M \in R^{n\times n}$, scalars $r_1 < r_2$ and vector function $\omega : [r_1, r_2] \to R^n$ such that the integrations concerned are well defined, then

$$\begin{aligned} &\left(\int_{r_1}^{r_2} \omega(s)\mathrm{d}s\right)^T M \left(\int_{r_1}^{r_2} \omega(s)\mathrm{d}s\right) \\ \le \quad &(r_2 - r_1)\int_{r_1}^{r_2} \omega^T(s) M\, \omega(s)\mathrm{d}s. \end{aligned}$$

Lemma 2 (see [1,24]) Let H, K and L be real matrices of appropriate dimensions with $\mathrm{K} > 0$. Then for any vectors x and y with appropriate dimensions, the following matrix inequality holds:

$$2x^T \mathrm{H}\,\mathrm{L}y \le x^T \mathrm{H}\,\mathrm{K}^{-1}\mathrm{H}^T x + y^T \mathrm{L}^T \mathrm{K}\,\mathrm{L}y.$$

Lemma 3 (see [23]) Assuming that function $g_j(s)$ is defined such that

$$0 \le \frac{g_j(s)}{s} \le \rho_j,$$

where $\rho_j > 0$, then the following inequality holds

$$\int_{\varsigma}^{\xi} (g_j(s) - g_j(\varsigma))\mathrm{d}s \le (\xi - \varsigma)(g_j(\xi) - g_j(\varsigma)).$$

Lemma 4 (see [4]) Let F,E, and Δ be real matrices of compatible dimensions with $\Delta = \mathrm{diag}\{\Delta_1, ..., \Delta_r\}, \Delta_i^T \Delta_i \le I$, $i = 1, ..., r$. Then, for any real matrix $\Lambda = \mathrm{diag}\{\lambda_1 I, ..., \lambda_r I\} > 0$, the following inequality holds:

$$F\Delta E + E^T \Delta^T F^T \le F\Lambda F^T + E^T \Lambda^{-1} E.$$

3. Globally exponential stability result of neural networks

First, we will present the exponential stability results for system (3) without uncertainties, that is

$$\begin{aligned} \dot{u}(t) = & -Cu(t) + Af(u(t)) + Bf(u(t-\tau(t))) \\ & + D\int_{t-\sigma(t)}^{t} g(u(s))\mathrm{d}s + E\dot{u}(t-\varpi). \end{aligned} \tag{7}$$

Before introducing the main results, following notations are defined for simplicity:

$$\begin{aligned} \Omega_{11} &= 2kP_{11} + 4kL_1(D_1 + D_2) - F_1C - CF_1^T + P_{13} + P_{13}^T \\ &\quad + e^{2k\bar{\tau}}(Q_{11} + R_1 + \bar{\tau}R_3) + 2L_1T_1L_1 + 2L_2T_3L_2 + X_1 + X_1^T, \\ \Omega_{12} &= (2kI - C)P_{12} - (1-\mu)(P_{13} - P_{14}) + P_{23}^T - X_1 + X_2^T, \\ \Omega_{13} &= -P_{14}, \Omega_{14} = P_{12}, \Omega_{15} = -CX_5^T, \Omega_{16} = F_1E, \\ \Omega_{17} &= F_1A - CF_2^T + e^{2k\bar{\tau}}Q_{12} + 2k(D_1 - D_2), \\ \Omega_{18} &= F_1B, \Omega_{1,10} = F_1D, \\ \Omega_{1,11} &= \bar{\tau}(2kI - C)P_{13} + \bar{\tau}P_{33}, \Omega_{1,12} = -\bar{\tau}X_1, \\ \Omega_{22} &= 2kP_{22} - (1-\mu)(Q_{11} + P_{23} + P_{23}^T - P_{24} - P_{24}^T) \end{aligned}$$

$$-X_2 - X_2^T + X_3 + X_3^T + 2L_1T_2L_1,$$
$$\Omega_{23} = -P_{24} - X_3 + X_4^T,\ \Omega_{24} = P_{22} - Q_{13},$$
$$\Omega_{26} = P_{12}^T E,\ \Omega_{27} = P_{12}^T A,$$
$$\Omega_{28} = P_{12}^T B - (1-\mu)Q_{12}, \Omega_{2,10} = P_{12}^T D,$$
$$\Omega_{2,11} = 2k\bar{\tau}P_{23} - \bar{\tau}(1-\mu)(P_{33} - P_{34}^T), \Omega_{2,12} = -\bar{\tau}X_2,$$
$$\Omega_{33} = -R_1 - X_4 - X_4^T, \Omega_{3,11} = -\bar{\tau}P_{34}^T,$$
$$\Omega_{44} = -Q_{33}, \Omega_{48} = -Q_{23}^T, \Omega_{4,11} = \bar{\tau}P_{23},$$
$$\Omega_{55} = -X_5 - X_5^T + e^{2k\bar{\tau}}(Q_{33} + \bar{\tau}R_2) + R_5,$$
$$\Omega_{56} = X_5E, \Omega_{57} = X_5A, \Omega_{58} = X_5B,$$
$$\Omega_{5,10} = X_5D, \Omega_{66} = -e^{-2k\varpi}R_5,$$
$$\Omega_{67} = E^TF_2^T, \Omega_{6,11} = \bar{\tau}E^TP_{13},$$
$$\Omega_{77} = F_2A + A^TF_2^T + e^{2k\bar{\tau}}Q_{22} - 2T_1,$$
$$\Omega_{78} = F_2B, \Omega_{7,10} = F_2D, \Omega_{7,11} = \bar{\tau}A^TP_{13},$$
$$\Omega_{88} = -(1-\mu)Q_{22} - 2T_2,$$
$$\Omega_{8,11} = \bar{\tau}B^TP_{13}, \Omega_{99} = \bar{\sigma}^2R_4 - 2T_3,$$
$$\Omega_{10,10} = -e^{-2k\bar{\sigma}}R_4, \Omega_{10,11} = \bar{\tau}D^TP_{13},$$
$$\Omega_{11,11} = 2k\bar{\tau}^2P_{33} - \bar{\tau}R_3, \Omega_{12,12} = -\bar{\tau}R_2,$$
$$\tilde{\Omega}_{1,11} = \bar{\tau}(2kI - C)P_{14} + \bar{\tau}P_{34}, \tilde{\Omega}_{1,12} = 0,$$
$$\tilde{\Omega}_{2,11} = 2k\bar{\tau}P_{24} - (1-\mu)(\bar{\tau}P_{34} - \bar{\tau}P_{44}),$$
$$\tilde{\Omega}_{2,12} = -\bar{\tau}X_3, \tilde{\Omega}_{3,11} = -\bar{\tau}P_{44}, \tilde{\Omega}_{3,12} = -\bar{\tau}X_4,$$
$$\tilde{\Omega}_{4,11} = \bar{\tau}P_{24},\ \tilde{\Omega}_{6,11} = \bar{\tau}E^TP_{14},$$
$$\tilde{\Omega}_{7,11} = \bar{\tau}A^TP_{14}, \tilde{\Omega}_{8,11} = \bar{\tau}B^TP_{14},$$
$$\tilde{\Omega}_{10,11} = \bar{\tau}D^TP_{14}, \tilde{\Omega}_{11,11} = 4k\bar{\tau}^2P_{44} - \bar{\tau}R_3,$$
$$F_1 = P_{11} + L_1(\Delta + \Lambda) + e^{2k\bar{\tau}}Q_{13},$$
$$F_2 = \Delta - \Lambda + e^{2k\bar{\tau}}Q_{23},$$
$$L_i = \mathrm{diag}\{l_{i1}, l_{i2}, \dots, l_{in}\}, i = 1, 2,$$

other parameters $\Omega_{ij}\ (i<j)$ are all equal to zero's and other parameters $\tilde{\Omega}_{ij}$ are defined as $\tilde{\Omega}_{ij} = \Omega_{ij}$.

Now, we present the stability results for system (7) with

$$0 \le \dot{\tau}(t) \le \eta < 1.$$

Theorem 1. Under the assumption (2), for given scalars $h > 0, \eta$, the equilibrium point of system (7) is globally exponentially stable with a convergence rate k for $0 \le \tau(t) \le \bar{\tau}, 0 \le \dot{\tau}(t) \le \eta < 1$ if there exist positive definite symmetric matrix $P = [P_{ij}]_{4\times4}$, nonnegative definite symmetric matrices $Q = [Q_{ij}]_{3\times3}, R_i (i = 1, \dots, 5)$, positive diagonal matrices $T_i (i = 1, 2, 3)$, $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \dots, \delta_n\}$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, real matrices $X_i (i = 1, \dots, 5)$ with compatible dimensions such that the following LMIs hold:

$$\begin{bmatrix} \Omega & \sqrt{2k}\cdot h\Phi^T \\ \sqrt{2k}\cdot h\Phi & -P_{44} \end{bmatrix} < 0, \tag{8}$$

$$\tilde{\Omega} = [\tilde{\Omega}_{ij}]_{12\times12} < 0, \tag{9}$$

where

$$\Omega = \left[\Omega_{ij}\right]_{12\times12},$$
$$\Phi = [0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad P_{34}^T\quad 0].$$

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(u(t)) = \sum_{i=1}^{4} V_i(u(t)) \tag{10}$$

with

$$V_1(u(t)) = e^{2kt}\eta^T(t)P\eta(t) + e^{2k\bar{\tau}}\int_{t-\tau(t)}^{t} e^{2ks}\xi^T(s)Q\xi(s)\mathrm{d}s,$$
$$V_2(u(t)) = 2e^{2kt}\sum_{i=1}^{n}\Big\{\delta_i\int_0^{u_i(t)}\big(f_i(s) + l_{1i}s\big)\mathrm{d}s + \lambda_i\int_0^{u_i(t)}\big(l_{1i}s - f_i(s)\big)\mathrm{d}s\Big\},$$
$$V_3(u(t)) = e^{2k\bar{\tau}}\int_{t-\bar{\tau}}^{t} e^{2ks}u^T(s)R_1u(s)\mathrm{d}s + e^{2k\bar{\tau}}\int_{t-\bar{\tau}}^{t}(s - t + \bar{\tau})e^{2ks}\dot{u}^T(s)R_2\dot{u}(s)\mathrm{d}s + e^{2k\bar{\tau}}\int_{t-\bar{\tau}}^{t}(s - t + \bar{\tau})e^{2ks}u^T(s)R_3u(s)\mathrm{d}s,$$
$$V_4(u(t)) = \bar{\sigma}\int_{t-\bar{\sigma}}^{t}(s - t + \bar{\sigma})e^{2ks}g^T(u(s))R_4g(u(s))\mathrm{d}s + \int_{t-\varpi}^{t} e^{2ks}\dot{u}^T(s)R_5\dot{u}(s)\mathrm{d}s,$$

where $\eta^T(t) = [u^T(t), u^T(t-\tau(t)), \int_{t-\tau(t)}^{t} u^T(s)\mathrm{d}s, \int_{t-h}^{t-\tau(t)} u^T(s)\mathrm{d}s],\quad \xi^T(s) = [u^T(s), f^T(u(s)), \dot{u}^T(s)].$

For convenience, we denote $u_\tau = u(t - \tau(t))$. The time derivative of functional (13) along the trajectories of system (7) is obtained as follows:

$$\dot{V}_1(u(t)) = e^{2kt}\Big\{2k\eta^T(t)P\eta(t) + 2\eta^T(t)P\dot{\eta}(t) + e^{2k\bar{\tau}}\xi^T(t)Q\xi(t) - e^{2k(\bar{\tau}-\tau(t))} \times (1 - \dot{\tau}(t))\xi^T(t-\tau(t))Q\xi(t-\tau(t))\Big\},$$
$$\dot{V}_2(u(t)) = e^{2kt}\Big\{4k\sum_{i=1}^{n}\Big[\delta_i\int_0^{u_i(t)}\big(f_i(s) + l_{1i}s\big)\mathrm{d}s + \lambda_i\int_0^{u_i(t)}\big(l_{1i}s - f_i(s)\big)\mathrm{d}s\Big] + 2\{f^T(u(t)) + u^T(t)L_1\}\Delta\dot{u}(t) + 2\{u^T(t)L_1 - f^T(u(t))\}\Lambda\dot{u}(t)\Big\},$$

$$\begin{aligned}\dot{V}_3(u(t)) = e^{2kt}\Big\{ & e^{2k\bar{\tau}}u^T(t)R_1u(t) - u^T(t-\bar{\tau})R_1u(t-\bar{\tau}) \\ & - e^{2k(\bar{\tau}-t)}\int_{t-\bar{\tau}}^{t} e^{2ks}\dot{u}^T(s)R_2\dot{u}(s)\mathrm{d}s \\ & + \bar{\tau}e^{2k\bar{\tau}}\left(\dot{u}^T(t)R_2\dot{u}(t) + u^T(t)R_3u(t)\right) \\ & - e^{2k(\bar{\tau}-t)}\int_{t-\bar{\tau}}^{t} e^{2ks}u^T(s)R_3u(s)\mathrm{d}s\Big\},\end{aligned}$$

$$\begin{aligned}\dot{V}_4(u(t)) = e^{2kt}\Big\{ & -\bar{\sigma}\int_{t-\bar{\sigma}}^{t} e^{2k(s-t)}g^T(u(s))R_4g(u(s))\mathrm{d}s \\ & + \bar{\sigma}^2 g^T(u(t))R_4g(u(t)) + \dot{u}^T(t)R_5\dot{u}(t) \\ & - e^{-2k\varpi}\dot{u}^T(t-\varpi)R_5\dot{u}(t-\varpi)\Big\}.\end{aligned}$$

It is easy to get the following inequalities by using Lemmas 1 and 2:

$$\begin{aligned} & \left(\int_{t-\tau(t)}^{t} u(s)\mathrm{d}s\right)^T P_{33}\int_{t-\tau(t)}^{t} u(s)\mathrm{d}s \\ & \le \tau(t)\int_{t-\tau(t)}^{t} u^T(s)P_{33}u(s)\mathrm{d}s, \qquad (11)\end{aligned}$$

$$\begin{aligned} & 2\left(\int_{t-\tau(t)}^{t} u(s)\mathrm{d}s\right)^T P_{34}\int_{t-\bar{\tau}}^{t-\tau(t)} u(s)\mathrm{d}s \\ & \le \left(\int_{t-\tau(t)}^{t} u(s)\mathrm{d}s\right)^T P_{34}P_{44}^{-1}P_{34}^T\int_{t-\tau(t)}^{t} u(s)\mathrm{d}s \\ & + \left(\int_{t-\bar{\tau}}^{t-\tau(t)} u(s)\mathrm{d}s\right)^T P_{44}\int_{t-\bar{\tau}}^{t-\tau(t)} u(s)\mathrm{d}s, \qquad (12)\end{aligned}$$

$$\begin{aligned} & \left(\int_{t-\bar{\tau}}^{t-\tau(t)} u(s)\mathrm{d}s\right)^T P_{44}\int_{t-\bar{\tau}}^{t-\tau(t)} u(s)\mathrm{d}s \\ & \le (\bar{\tau}-\tau(t))\int_{t-\bar{\tau}}^{t-\tau(t)} u^T(s)P_{44}u(s)\mathrm{d}s, \qquad (13)\end{aligned}$$

$$\begin{aligned} & \left(\int_{t-\tau(t)}^{t} u(s)\mathrm{d}s\right)^T P_{34}P_{44}^{-1}P_{34}^T\int_{t-\tau(t)}^{t} u(s)\mathrm{d}s \\ & \le \tau(t)\int_{t-\tau(t)}^{t} u^T(s)P_{34}P_{44}^{-1}P_{34}^T u(s)\mathrm{d}s. \qquad (14)\end{aligned}$$

Further, from inequality (4) and Lemma 3 we have

$$\begin{aligned} & \sum_{i=1}^{n}\delta_i\int_0^{u_i(t)}\left(f_i(s)+l_{1i}s\right)\mathrm{d}s \\ & \le \{f^T(u(t)) + u^T(t)L_1\}\Delta u(t), \qquad (15)\end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^{n}\lambda_i\int_0^{u_i(t)}\left(l_{1i}s - f_i(s)\right)\mathrm{d}s \\ & \le \{u^T(t)L_1 - f^T(u(t))\}\Lambda u(t). \qquad (16)\end{aligned}$$

In addition, the following inequality holds from Lemma 1:

$$\begin{aligned} - & \bar{\sigma}\int_{t-\bar{\sigma}}^{t} g^T(u(s))R_4g(u(s))\mathrm{d}s \\ \le & -\left(\int_{t-\bar{\sigma}}^{t} g(u(s))\mathrm{d}s\right)^T R_4\int_{t-\bar{\sigma}}^{t} g(u(s))\mathrm{d}s \\ \le & -\left(\int_{t-\sigma(t)}^{t} g(u(s))\mathrm{d}s\right)^T R_4\int_{t-\sigma(t)}^{t} g(u(s))\mathrm{d}s. \qquad (17)\end{aligned}$$

On the other hand, one can infer from inequality (4) that the following matrix inequalities hold for any positive diagonal matrices $T_i(i=1,2,3)$ with compatible dimensions

$$0 \le 2e^{2kt}\left\{u^T(t)L_1T_1L_1u(t) - f^T(u(t))T_1f(u(t))\right\}, \qquad (18)$$

$$0 \le 2e^{2kt}\left\{u_\tau^T L_1T_2L_1u_\tau - f^T(u_\tau)T_2f(u_\tau)\right\}, \qquad (19)$$

$$0 \le 2e^{2kt}\left\{u^T(t)L_2T_3L_2u(t) - g^T(u(t))T_3g(u(t))\right\}. \qquad (20)$$

Based on Leibniz-Newton formula, for any real matrices $X_i(i=1,\dots,4)$ with compatible dimensions, we get

$$\begin{aligned} 0 = 2e^{2kt}\{ & u^T(t)X_1 + u_\tau^T X_2\} \\ & \times\left\{u(t) - u_\tau - \int_{t-\tau(t)}^{t}\dot{u}(s)\mathrm{d}s\right\}, \qquad (21)\end{aligned}$$

$$\begin{aligned} 0 = 2e^{2kt}\{ & u_\tau^T X_3 + u^T(t-\bar{\tau})X_4\} \\ & \times\left\{u_\tau - u(t-\bar{\tau}) - \int_{t-\bar{\tau}}^{t-\tau(t)}\dot{u}(s)\mathrm{d}s\right\}. \qquad (22)\end{aligned}$$

To get less conservative criterion, we introduce the following equality for any real matrix X_5 with compatible dimension

$$\begin{aligned} 0 = 2\dot{u}^T(t)X_5\Big\{ & -\dot{u}(t) - Cu(t) + Af(u(t)) + Bf(u_\tau) \\ & + D\int_{t-\sigma(t)}^{t} g(u(s))\mathrm{d}s + E\dot{u}(t-\varpi)\Big\}. \qquad (23)\end{aligned}$$

From (10)-(23), we obtain

$$\begin{aligned}\dot{V}(u(t)) \le \ & \frac{1}{\bar{\tau}}e^{2kt}\Big(\int_{t-\bar{\tau}}^{t-\tau(t)}\zeta^T(t,s)\tilde{\Omega}\zeta(t,s)\mathrm{d}s \\ & + \int_{t-\tau(t)}^{t}\zeta^T(t,s)(\Omega + \bar{\tau}^2\Phi^T P_{44}\Phi)\zeta(t,s)\mathrm{d}s\Big),\end{aligned}$$

where

$$\begin{aligned}\zeta^T(t,s) = [\ & u^T(t), u_\tau^T, u^T(t-\bar{\tau}), (1-\dot{\tau}(t))\dot{u}^T(t-\tau(t)), \\ & \dot{u}^T(t), \dot{u}^T(t-\varpi),\ f^T(u(t)),\ f^T(u_\tau), \\ & g^T(u(t)), \int_{t-\sigma(t)}^{t} g^T(u(s))\mathrm{d}s, u^T(s),\ \dot{u}^T(s)\ ].\end{aligned}$$

Thus $\dot{V}(u(t)) < 0$ holds if inequalities (8) and (9) are true. Furthermore, following the similar line in [23,24], from Lemma 2 we have

$$V(u(0)) \le M\Big(\sup_{-\max\{\bar{\tau},\bar{\sigma},\varpi\}\le\theta\le 0}(\|u(\theta)\|, \|\dot{u}(\theta)\|)\Big)^2,$$

where

$$\begin{aligned} M = \ & 4(2+\bar{\tau})\lambda_M(P) + 2\lambda_M(D+\Lambda)\lambda_M(L_1) + \varpi\lambda_M(R_5) \\ & + e^{2k\bar{\tau}}\left\{3\lambda_M(Q)[1+\bar{\tau}+\lambda_M(L_1^2)] + \bar{\tau}\lambda_M(R_1)\right. \\ & \left.+\frac{1}{2}\bar{\tau}^2[\lambda_M(R_2)+\lambda_M(R_3)]\right\} + \frac{1}{2}\bar{\sigma}^3\lambda_M(L_2^2)\lambda_M(R_4).\end{aligned}$$

Meanwhile $V(u(t)) \ge e^{2kt}\|\phi(t) - z^*\|^2\lambda_m(P_{11})$, by Lyapunov stability theory, the proof of Theorem 1 is completed.

Remark 1. It is easy to see that the derivatives of $\eta^T(t)P\eta(t)$ and $\int_{t-\tau(t)}^{t}\xi^T(s)Q\xi(s)\mathrm{d}s$ have some terms containing $1-\dot{\tau}(t)$. In order to absorb some $1-\dot{\tau}(t)$, we introduce $(1-\dot{\tau}(t))\dot{u}^T(t-\tau(t))$ in $\zeta(t)$ but not $\dot{u}^T(t-\tau(t))$, so Ω and $\tilde{\Omega}$ contains fewer $1-\dot{\tau}(t)$, which leads to a more effective results.

Remark 2. In Theorem 1, if we set $P_{ij}=0(i+j>2)$, $Q=0$, by deleting $(1-\dot{\tau}(t))\dot{u}^T(t-\tau(t))$ from $\zeta^T(t,s)$, we can employ this criterion to analyze the stability of neural network when $\dot{\tau}(t)$ is unknown or $\tau(t)$ is not differentiable.

Remark 3. If $f_i(x)=g_i(x)(i=1,...,n)$ in neural network (7), by applying the same functional as in Theorem 1 and deleting $g^T(u(t))$ from $\zeta^T(t,s)$, similar to above proof and Remark 2, from (10)-(19) and (21)-(23) we can derive a criterion to analyze the stability of neural networks (7) with $\mathrm{f}(\mathrm{x})=\mathrm{g}(\mathrm{x})$.

Remark 4. If *E=0* in neural network (7), by setting $R_5=0$ in Theorem 1 and deleting $\dot{u}^T(t-\varpi)$ from $\zeta^T(t,s)$, similar to above proof and Remark 2, we can derive a criterion to analyze the stability of mixed-delay neural networks (7) with *E=0*.

Remark 5. If *D=0* in neural network (7), by setting $R_4=0$ in Theorem 1 and deleting $g^T(u(t))$, $\int_{t-\sigma(t)}^{t} g^T(u(s))\mathrm{d}s$ from $\zeta^T(t,s)$, similar to above proof and Remark 2, from (10)-(16) and (18)-(23) we can derive a criterion to analyze the stability of neutral-type neural networks (7) with *D=0*.

4. Robust exponential stability results of uncertain delayed neural network

On the basis of the results of Theorem 1, from Lemma 4 it is easy to obtain the following conclusion about the robust stability condition for system (1) with norm-bounded uncertainties satisfying (5) and (6).

Theorem 2. Under the assumption (2), for given scalars $h>0,\eta$, the equilibrium point of system (1) is globally robust exponentially stable with a convergence rate *k* for $0\le\tau(t)\le\overline{\tau}, 0\le\dot{\tau}(t)\le\eta<1$ if there exist constant $\varepsilon>0$, positive definite symmetric matrix $P=[P_{ij}]_{4\times4}$, non-negative definite symmetric matrices $Q=[Q_{ij}]_{3\times3}$, $R_i(i=1,...,5)$, positive diagonal matrices $T_i(i=1,2,3)$, Δ,Λ, real matrices $X_i(i=1,...,5)$ with compatible dimensions such that the following LMIs hold:

$$\begin{bmatrix} \Omega+\varepsilon\Theta^T\Theta & \sqrt{2k}\cdot h\Phi^T & \Psi^T H \\ \sqrt{2k}\cdot h\Phi & -P_{44} & 0 \\ H^T\Psi & 0 & -\varepsilon I \end{bmatrix}<0, \tag{24}$$

$$\begin{bmatrix} \tilde{\Omega}+\varepsilon\Theta^T\Theta & \tilde{\Psi}^T H \\ H^T\tilde{\Psi} & -\varepsilon I \end{bmatrix}<0, \tag{25}$$

where

$$\Theta = [-G_0\ \ 0\ \ 0\ \ 0\ 0\ \ G_4\ \ G_1\ \ G_2\ \ 0\ \ G_3\ \ 0\ \ 0],$$
$$\Psi = [F_0^T\ \ P_{12}\ \ 0\ \ 0\ \ X_5^T\ \ 0\ \ F_1^T\ \ 0\ 0\ 0\ \ \overline{\tau}P_{13}^T\ \ 0],$$
$$\tilde{\Psi} = [F_0^T\ \ P_{12}\ \ 0\ \ 0\ \ X_5^T\ \ 0\ F_1^T\ \ 0\ \ 0\ \ 0\ \ \overline{\tau}P_{14}^T\ \ 0],$$

and other parameters are defined in Theorem 1.

Remark 6. Similar to Remark 3, by setting $P_{ij}=0(i+j>2)$, *Q=0*, we can employ the criterion of Theorem 2 to analyze the robust stability of neural network (1) when $\dot{\tau}(t)$ is unknown or $\tau(t)$ is not differentiable.

Remark 7. If $f_i(x)=g_i(x)(i=1,...,n)$ in neural network (1), similar to Remark 3, from Theorem 2 and Remark 6 we can derive a criterion to analyze the robust stability of neural networks (1) with *f(x)=g(x)*.

Remark 8. If *E=0* in neural network (1), by setting $R_5=0$ in Theorem 2, similar to Remarks 4,6, we can derive a criterion to analyze the robust stability of mixed-delay neural networks (1) with *E=0*.

Remark 9. If *D=0* in neural network (1), by setting $R_4=0$ in Theorem 2, similar to Remarks 5,6 we can derive a criterion to analyze the robust stability of neutral-type neural networks (1) with *D=0*.

5. Comparison and Illustrative Examples

In this section, we provide four numerical examples to demonstrate the effectiveness and less conservativeness of our delay-dependent stability criteria over some recent results in the literature.

Example 1. Consider system (7) with

$$C=\mathrm{diag}\{3,3.4\},\ J=[-0.9\quad -1.5]^T$$
$$A=\begin{bmatrix}-0.9 & -1.5\\ -1.2 & 1\end{bmatrix},\ B=\begin{bmatrix}0.8 & 0.6\\ -0.1 & 0.5\end{bmatrix},$$
$$D=\begin{bmatrix}-0.2 & 0.1\\ -0.1 & 0.1\end{bmatrix},\ E=\begin{bmatrix}0.5 & 0.2\\ -0.1 & 0.6\end{bmatrix},$$
$$f_i(x)=g_i(x)(i=1,2),\ L_1=I.$$

Obviously, the stability of this model can't be ascertained by using the conditions in [5]–[7], [11], [13], [15], [16]. However, if we set time delays be constants $\tau(t)=\sigma(t)=\overline{\tau}$ and $\varpi=1,k=0.2$, by Remark 3 of Theorem 1 we can conclude that, the equilibrium point of this system is exponential stable for any time delay with $\overline{\tau}\le1.5976$.

Example 2. Consider system (7) with

$$C=\mathrm{diag}\{2.3,3.4,2.5\},\ E=0,$$

$$A=\begin{bmatrix}0.9 & -1.5 & 0.1\\ -1.2 & 1 & 0.2\\ 0.2 & 0.3 & 0.8\end{bmatrix},$$
$$B=\begin{bmatrix}0.8 & 0.6 & 0.2\\ 0.5 & 0.7 & 0.1\\ 0.2 & 0.1 & 0.5\end{bmatrix},$$
$$D=\begin{bmatrix}0.3 & 0.2 & 0.1\\ 0.1 & 0.2 & 0.1\\ 0.1 & 0.1 & 0.2\end{bmatrix},$$
$$f_i(x)=g_i(x)(i=1,2,3),\ L_1=0.2I.$$

For this model, if we set exponential convergence rate k be fixed as 0 (this means the asymptotic stability), the maximal upper bounds of time delays for various η 's from Remark 4 of Theorem 1 in this paper and those in [13] are listed in Table I. It is clear that the results in this paper are much better than those in [13].

Table I Calculated maximal upper bounds of time delays for various η of Example 2 with *k=0*

methods	$\eta=0$	$\eta=0.8$	unknown η
[13]	2.8384	2.8384	2.8384$
This paper	23.7800	20.5183	6.9332

Example 3. Consider system (7) with constant delays $\tau(t)=\varpi$, and

$$C=\mathrm{diag}\{2.7644, 1.0185, 10.2716\},\ D=0,$$
$$A=\begin{bmatrix}0.2651 & -3.1608 & -2.0491\\ 3.1859 & -0.1573 & -2.4687\\ 2.0368 & -1.3633 & 0.5776\end{bmatrix},$$
$$B=\begin{bmatrix}-0.7727 & -0.8370 & 3.8019\\ 0.1004 & 0.6677 & -2.4431\\ -0.6622 & 1.3109 & -1.8407\end{bmatrix},$$
$$E=\begin{bmatrix}0.2076 & 0.0631 & 0.3915\\ -0.0780 & 0.3106 & 0.1009\\ -0.2763 & 0.1416 & 0.3729\end{bmatrix},$$
$$f_i(x)=g_i(x)(i=1,2,3),$$
$$L_1=\mathrm{diag}\{0.1019, 0.3419, 0.0633\}.$$

This model was studied in [11,15]. Ref. [15] illustrated that the maximum bound of delays is 1.0344. Let *m=3* in [11], the authors obtained the upper bound of delay is 82. However by using our Remark 5 to this example, we can obtain the system is feasible for any $\varpi>0$. It means that the system is delay-independent stable, which shows that our criteria are less conservative than [11,15].

Example 4. Consider system (3) with

$$C=\mathrm{diag}\{2,3\},$$
$$A=\begin{bmatrix}0.5 & -0.1\\ -0.2 & -0.3\end{bmatrix},\ B=\begin{bmatrix}0.2 & -0.4\\ 0.1 & 0.2\end{bmatrix},$$
$$D=\begin{bmatrix}0.4 & 0.3\\ 0.1 & 0.2\end{bmatrix},\ E=\begin{bmatrix}0.5 & 0.2\\ -0.1 & 0.6\end{bmatrix},$$
$$H=I,\ G_0=\begin{bmatrix}0 & 0\\ 0.2 & 0.2\end{bmatrix},$$
$$G_1=\begin{bmatrix}0 & 0\\ 0.1 & 0.1\end{bmatrix},\ G_2=\begin{bmatrix}0 & 0\\ 0.3 & 0.3\end{bmatrix},$$
$$G_3=\begin{bmatrix}0 & 0\\ 0.2 & 0.2\end{bmatrix},\ G_4=\begin{bmatrix}0 & 0\\ 0.1 & 0.2\end{bmatrix},$$
$$f_i(x)=g_i(x)(i=1,2),\ L_1=I.$$

Obviously, none of the criteria in [2,8,14,21–23] can be applied to verify the stability of this system.

However, if we set exponential convergence rate of k be fixed as 0, from Remark 7 of Theorem 2 we can confirm that the equilibrium point of this system is robust exponential stable for any constant time delays with $\tau(t)=\sigma(t)=\overline{\tau}\le 1.8444, \varpi>0$.

Therefore, we can say that for these four systems the results in this paper are much effective and less conservative than those in [2,5–8,11,13–16,21–23].

6. Conclusions

In this paper we have investigated the global robust stability problem of uncertain neutral-type neural networks with discrete and distributed delays. By employing new Lyapunov Krasovskii functional, we proposed several novel stability criteria for the considered neural networks. The obtained results are all in the form of LMI, which can be easily optimized. Finally, four examples are given to show the superiority of our proposed stability conditions to some existing ones.

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