Novel robust stability criteria of neutral-type bidirectional associative memory neural networks

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Abstract

The existence, uniqueness and global robust exponential stability is analyzed for a class of uncertain neutral-type bidirectional associative memory (BAM) neural networks with time-varying delays. Without assuming the boundedness of the activation functions, by constructing a novel class of augmented Lyapunov-Krasovskii functional, new relaxed delay-dependent stability criteria of the unique equilibrium point are presented in terms of linear matrix inequalities (LMIs). Following the idea of convex combination and free-weighting matrices method, less conservative results are obtained. Two examples are given to illustrate the effectiveness of our proposed conditions.

Keywords: *Global robust exponential stability, globally exponential stability, linear matrix inequality(LMI), neutral-type, bidirectional associative memory (BAM) neural network.*

1. Introduction

Bidirectional associative memory (BAM) neural networks, which were proposed by Kosko in [10,11], generalized the single-layer autoassociative Hebbian correlator to a two-layer pattern-matched heteroassociative circuits. There are many applications for BAM neural networks such as pattern recognition, artificial intelligence, solving optimization problem and automatic control engineering. Therefore, the BAM neural networks have been one of the most interesting research topics and have attracted the attention of many researchers. Up to now, many important results on the stability of BAM neural networks have been reported in the literature, see e.g. [2]–[4], [7], [8], [12], [15]–[19] and references therein.

On the other hand, the stability of a neural network may often be destroyed by its unavoidable uncertainties due to the existence of modeling errors, external disturbance and parameter fluctuation in the applications and designs of neural networks. Therefore, the robust stability analysis of neural networks has gained much research attention [15]–[17], [19], [24]–[27]. However, the existence of time delays in these DNN models indicates that time delays are dependent on the past state. In fact, many practical delay systems can be modeled as differential systems of neutral type, whose differential expression concludes not only the derivative term of the current state but also concludes the derivative term of the past state, such as partial element equivalent circuits and transmission lines in electrical engineering, controlled constrained manipulators in mechanical engineering, population dynamics system and so on (see [13]). It is natural and important that systems should contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions. There have been many results about neutral-type cellular neural networks, but to the best of our knowledge, few researchers studied the stability for BAM neural networks which is described by nonlinear delay differential equations of the neutral type. By using Jensen integral inequality, Liu et al. [12] recently proposed stability conditions of neutral-type BAM neural networks with constant delays which are expressed in terms of LMIs. Based on an inequality, Park et al. [14] obtained a condition of globally asymptotic stability for such systems also with constant time delays. Up to now, there are no results about such systems with time varying delays.

Motivated by the preceding discussions, the aim of this paper is to relax the constraint on the boundedness of the activation function, and study the existence, uniqueness and global robust exponential stability for uncertain neutral-type BAM neural networks with time-varying delays. The authors introduce a novel form of augmented Lyapunov-Krasovskii functional that takes into account new terms $e^{2kt}u^T(t-\sigma(t))P_{22}u(t-\sigma(t))$ and $e^{2kt}v^T(t-\tau(t)) \times Q_{22}v(t-\tau(t))$, whose derivatives are directly coupled with both neutral and retarded systems. The proposed Lyapunov functional also includes the terms of cross products $e^{2kt}u^T(t)P_{12}u(t-\sigma(t))$, $e^{2kt}v^T(t)Q_{12}v(t-\tau(t))$, and some integral terms of cross products, such as $\int_{t-\sigma(t)}^{t} e^{2ks}u^T(s)R_{12}g(u(s))\mathrm{d}s$, $\int_{t-\tau(t)}^{t} e^{2ks}v^T(s)S_{12}f(v(s))\mathrm{d}s$,

which are not considered in previous results. Following the idea of convex combination and free-weighting matrices method [9], we derive several new sufficient conditions for the global exponential stability of BAM neural networks with time-varying delays. The derived conditions are expressed in terms of linear matrix inequalities (LMIs), which can be checked numerically very efficiently via the LMI toolbox. Some comparisons between the obtained results in this paper and previous results are made by two illustrative examples.

The rest of this paper is organized as follows. In Section II, problem formulation and preliminaries are given. In Sections III,IV, new delay-dependent conditions are established for the existence, uniqueness and exponential stability. In Section V, the new stability conditions are extended to neural networks with norm-bounded uncer-tainties. Section VI provides two illustrative examples. Finally, some conclusions are drawn in Section VII.

2. Problem description

Considering the following neutral-type neural networks with time-varying delays:

$$\begin{cases} \dot{x}(t)= & -\bar{A}_1 x(t)+\bar{B}_1\tilde{f}(y(t))+\bar{C}_1\tilde{f}(y(t-\tau(t))) \\ & +\bar{D}_1\dot{x}(t-\sigma(t))+J, \\ \dot{y}(t)= & -\bar{A}_2 y(t)+\bar{B}_2\tilde{g}(x(t))+\bar{C}_2\tilde{g}(x(t-\sigma(t))) \\ & +\bar{D}_2\dot{y}(t-\tau(t))+E, \end{cases} \quad (1)$$

where $x(t)=(x_1(t),x_2(t),...,x_n(t))^T$ and $y(t)=(y_1(t), y_2(t),...,y_m(t))^T$ are the neural state vectors, J,E are the constant external input vectors. $\bar{A}_1=A_1+\Delta A_1(t)$, $\bar{B}_1=B_1+\Delta B_1(t)$, $\bar{C}_1=C_1+\Delta C_1(t)$, $\bar{D}_1=D_1+\Delta D_1(t)$, $\bar{A}_2=A_2+\Delta A_2(t)$, $\bar{B}_2=B_2+\Delta B_2(t)$, $\bar{C}_2=C_2+\Delta C_2(t)$, $\bar{D}_2=D_2+\Delta D_2(t)$. $A_1=\text{diag}\{a_{11},a_{12},...,a_{1n}\}$ and $A_2=\text{diag}\{a_{21},a_{22},...,a_{2m}\}$ are positive diagonal matrices, $B_1=(b_{1ij})_{n\times m}$, $C_1=(c_{1ij})_{n\times m}$, $D_1=(d_{1ij})_{n\times n}$, $B_2=(b_{2ij})_{m\times n}$, $C_2=(c_{2ij})_{m\times n}$, $D_2=(d_{2ij})_{m\times m}$ are known constant matrices, $\Delta A_1(t),\Delta B_1(t),\Delta C_1(t)$, $\Delta D_1(t)$, $\Delta A_2(t),\Delta B_2(t),\Delta C_2(t),\Delta D_2(t)$ are parametric uncertainties, $0\le\tau(t)\le\bar{\tau}$, $0\le\sigma(t)\le\bar{\sigma}$ are the time-varying delays, where $\bar{\tau},\bar{\sigma}$ are positive constants. $\tilde{f}(x(t))=(\tilde{f}_1(y_1(t)), \tilde{f}_2(y_2(t)),...,\tilde{f}_m(y_m(t)))^T$, $\tilde{g}(x(t))=(\tilde{g}_1(x_1(t)),...,\tilde{g}_n(x_n(t)))^T$ denote the neural activation functions. It is assumed that there exist constants $l_{1i},l_{2i},l_{3j},l_{4j}$ such that

$$l_{1i}\le\frac{\tilde{f}_i(s_1)-\tilde{f}_i(s_2)}{s_1-s_2}\le l_{2i},$$

$$l_{3j}\le\frac{\tilde{g}_j(s_1)-\tilde{g}_j(s_2)}{s_1-s_2}\le l_{4j},$$

for any $s_1,s_2\in\mathrm{R}$, $s_1\ne s_2, i=1,2,...,m;$ $j=1,2,...,n.$

Moreover, we assume that the initial condition of neural networks (1) has the form

$$x_j(t)=\phi_j(t),\ y_i(t)=\varphi_i(t),\quad t\in[-\max\{\bar{\sigma},\bar{\tau}\},0]$$

where $\phi_j(t),\varphi_i(t)(i=1,2,...,m;\ j=1,2,...,n)$ are continuous functions.

Throughout this paper, let $\|y\|$ denote the Euclidean norm of a vector $y\in\mathrm{R}^n$, $W^T,W^{-1},\lambda_M(W),\lambda_m(W)$ and $\|W\|=\sqrt{\lambda_M(W^TW)}$ denote the transpose, the inverse, the largest eigenvalue, the smallest eigenvalue, and the spectral norm of a square matrix W, respectively. Let $W>0(<0)$ denote a positive (negative) definite symmetric matrix, I denote an identity matrix with compatible dimension.

The definition of exponential stability is now given.

Definition 1([20]) The neural network (1) is said to be globally exponentially stable if there exist constants $k\ge 0$ and $M>1$ such that

$$\|x(t)\|+\|y(t)\|\le M\Big(\sup_{-\bar{\sigma}\le\theta\le 0}\{\|x(\theta)\|,\|\dot{x}(\theta)\|\}+\sup_{-\bar{\tau}\le\theta\le 0}\{\|y(\theta)\|,\|\dot{y}(\theta)\|\}\Big)e^{-kt},$$

where k is called the exponential convergence rate.

Clearly, the equilibrium point of neural network (1) is robust stable if and only if the zero solution of neural network (4) is robust stable.

The time-varying uncertain matrices are defined by:

$$\begin{cases} [\Delta A_1(t)\quad \Delta B_1(t)\quad \Delta C_1(t)\quad \Delta D_1(t)] \\ \qquad =H_0E_0(t)[G_1\quad G_2\quad G_3\quad G_4], \\ [\Delta A_2(t)\quad \Delta B_2(t)\quad \Delta C_2(t)\quad \Delta D_2(t)] \\ \qquad =H_1E_1(t)[G_5\quad G_6\quad G_7\quad G_8], \end{cases} \quad (4)$$

where $H_0,H_1,G_i(i=1,...,8)$ are known real constant matrices with appropriate dimensions. $E_0(t),E_1(t)$ are unknown time-varying matrices satisfying

$$E_0^T(t)E_0(t)\le I,\ E_1^T(t)E_1(t)\le I. \quad (5)$$

In order to obtain the main results, we need the following lemmas.

Lemma 1 (see [1,23]) Let X,Y and P be real matrices of appropriate dimensions with $P>0$. Then for any positive scalar ε the following matrix inequality holds:

$$X^TY+Y^TX\le\varepsilon^{-1}X^TP^{-1}X+\varepsilon Y^TPY.$$

Lemma 2(see [27]) Continuous map $T(z):\mathrm{R}^n\to\mathrm{R}^n$ is homeomorphic if $T(z)$ is injective and $\lim_{\|z\|\to\infty}\|T(z)\|=\infty.$

3. Existence and uniqueness of the equilibrium point

Firstly, we consider neural network (1) without uncertainties, that is

$$\begin{cases} \dot{x}(t) = & -A_1 x(t) + B_1 \tilde{f}(y(t)) + C_1 \tilde{f}(y(t-\tau(t))) \\ & +D_1 \dot{x}(t-\sigma(t)) + J, \\ \dot{y}(t) = & -A_2 y(t) + B_2 \tilde{g}(x(t)) + C_2 \tilde{g}(x(t-\sigma(t))) \\ & +D_2 \dot{y}(t-\tau(t)) + E. \end{cases} \quad (6)$$

In order to study the existence and uniqueness of the equilibrium point, we define map $\mathrm{H}(w)$ as $\mathrm{H}(w) = [\mathrm{H}_1^T(w)\ \mathrm{H}_2^T(w)]^T$, where $w = [x^T\ \ y^T]^T$ and

$$\mathrm{H}_1(w) = -A_1 x + (B_1 + C_1)\tilde{f}(y) + J,$$
$$\mathrm{H}_2(w) = -A_2 y + (B_2 + C_2)\tilde{g}(x) + E.$$

Theorem 1. Under assumptions (2),(3) and $0 \le \tau(t) \le \bar{\tau}$, $0 \le \sigma(t) \le \bar{\sigma}$, $0 \le \dot{\tau}(t) \le \eta_1 < 1$, $0 \le \dot{\sigma}(t) \le \eta_2 < 1$, given a constant $k \ge 0$, suppose that there exist positive definite symmetric matrices $P = [P_{ij}]_{2\times 2}, Q = [Q_{ij}]_{2\times 2}$, non-negative definite symmetric matrices $R = [R_{ij}]_{3\times 3}$, $S = [S_{ij}]_{3\times 3}$, $U_i, Z_i (i = 1,2)$, positive diagonal matrices T_j, real matrices $X_j, Y_j (j = 1,2,3,4)$ with compatible dimensions such that the following LMIs hold (*i,j=1,2*):

$$\begin{bmatrix} \Omega & \Phi_i^T & \Psi_j^T \\ \Phi_i & -e^{-2k\bar{\sigma}} U_1 & 0 \\ \Psi_j & 0 & -e^{-2k\bar{\tau}} U_2 \end{bmatrix} < 0, \quad (7)$$

where

$$\Omega = [\ \Omega_{ij}\ ]_{14\times 14},$$
$$\Omega_{11} = 2kP_{11} - (P_{11} + R_{13})A_1 - A_1(P_{11} + R_{13}^T) + R_{11} + X_1 + X_1^T - 2L_3 T_3 L_4,$$
$$\Omega_{12} = (2kI - A_1)P_{12} - X_1^T + X_2,$$
$$\Omega_{14} = R_{12} - A_1 R_{23}^T + (L_3 + L_4)T_3,$$
$$\Omega_{16} = (P_{11} + R_{13})D_1 + (1-\eta_2)P_{12},\ \Omega_{17} = -A_1 Z_1^T,$$
$$\Omega_{1,11} = (P_{11} + R_{13})B_1,\ \Omega_{1,12} = (P_{11} + R_{13})C_1,$$
$$\Omega_{22} = 2kP_{22} - (1-\eta_2)e^{-2k\bar{\sigma}} R_{11} - X_2 - X_2^T + X_3 + X_3^T - 2L_3 T_4 L_4,\quad \Omega_{23} = -X_3^T + X_4,$$
$$\Omega_{25} = -(1-\eta_2)e^{-2k\bar{\sigma}} R_{12} + (L_3 + L_4)T_4,$$
$$\Omega_{26} = P_{12}^T D_1 + (1-\eta_2)P_{22} - (1-\eta_2)e^{-2k\bar{\sigma}} R_{13},$$
$$\Omega_{2,11} = P_{12}^T B_1,\ \Omega_{2,12} = P_{12}^T C_1,\ \Omega_{33} = -X_4 - X_4^T,$$
$$\Omega_{44} = R_{22} - 2T_3,\quad \Omega_{46} = R_{23} D_1,$$
$$\Omega_{48} = B_2^T(Q_{11} + S_{13}^T),\quad \Omega_{49} = B_2^T Q_{12},$$
$$\Omega_{4,11} = R_{23} B_1 + B_2^T S_{23}^T,$$
$$\Omega_{4,12} = R_{23} C_1,\quad \Omega_{4,14} = B_2^T Z_2^T,$$
$$\Omega_{55} = -(1-\eta_2)e^{-2k\bar{\sigma}} R_{22} - 2T_4,$$
$$\Omega_{56} = -(1-\eta_2)e^{-2k\bar{\sigma}} R_{23},\ \Omega_{58} = C_2^T(Q_{11} + S_{13}^T),$$
$$\Omega_{59} = C_2^T Q_{12},\ \ \Omega_{5,11} = C_2^T S_{23}^T,\ \Omega_{5,14} = C_2^T Z_2^T,$$
$$\Omega_{66} = -(1-\eta_2)e^{-2k\bar{\sigma}} R_{33},\ \Omega_{67} = D_1^T Z_1^T,$$
$$\Omega_{77} = -Z_1 - Z_1^T + R_{33} + \bar{\sigma}^2 U_1,$$
$$\Omega_{7,11} = Z_1 B_1,\ \ \Omega_{7,12} = Z_1 C_1,$$
$$\Omega_{88} = 2kQ_{11} - (Q_{11} + S_{13})A_2 - A_2(Q_{11} + S_{13}^T) + S_{11} + Y_1 + Y_1^T - 2L_1 T_1 L_2,$$
$$\Omega_{89} = (2kI - A_2)Q_{12} - Y_1^T + Y_2,$$
$$\Omega_{8,11} = S_{12} - A_2 S_{23}^T + (L_1 + L_2)T_1,$$
$$\Omega_{8,13} = (Q_{11} + S_{13})D_2 + (1-\eta_1)Q_{12},\ \Omega_{8,14} = -A_2^T Z_2^T,$$
$$\Omega_{99} = 2kQ_{22} - (1-\eta_1)e^{-2k\bar{\tau}} S_{11} - Y_2 - Y_2^T + Y_3 + Y_3^T - 2L_1 T_2 L_2,\quad \Omega_{9,10} = -Y_3^T + Y_4,$$
$$\Omega_{9,12} = -(1-\eta_1)e^{-2k\bar{\tau}} S_{12} + (L_1 + L_2)T_2,$$
$$\Omega_{9,13} = Q_{12}^T D_2 + (1-\eta_1)Q_{22} - (1-\eta_1)e^{-2k\bar{\tau}} S_{13},$$
$$\Omega_{10,10} = -Y_4 - Y_4^T,\ \Omega_{11,11} = S_{22} - 2T_1,$$
$$\Omega_{11,13} = S_{23} D_2,\ \Omega_{12,12} = -(1-\eta_1)e^{-2k\bar{\tau}} S_{22} - 2T_2,$$
$$\Omega_{12,13} = -(1-\eta_1)e^{-2k\bar{\tau}} S_{23},$$
$$\Omega_{13,13} = -(1-\eta_1)e^{-2k\bar{\tau}} S_{33},\ \Omega_{13,14} = D_2^T Z_2^T,$$
$$\Omega_{14,14} = -Z_2 - Z_2^T + S_{33} + \bar{\tau}^2 U_2,$$
$$\Phi_1 = [\ X_1\ \ X_2\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ],$$
$$\Phi_2 = [\ 0\ X_3\ \ X_4\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ],$$
$$\Psi_1 = [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ Y_1\ Y_2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ],$$
$$\Psi_2 = [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ Y_3\ \ Y_4\ \ 0\ \ 0\ \ 0\ \ 0\ ],$$
$$L_i = \mathrm{diag}\{l_{i1}, l_{i2}, \ldots, l_{in}\},\ i = 1,2,3,4,$$

and other parameters $\Omega_{ij}(i < j)$ are all equal to zero's, then neural network (6) have a unique equilibrium point.

Proof. As done in [6], we will firstly prove that $\mathrm{H}(w) \ne \mathrm{H}(\bar{w})$ for any $w \ne \bar{w},\ w, \bar{w} \in \mathrm{R}^{n+m}$.

Now suppose $\mathrm{H}(w) = \mathrm{H}(\bar{w})$, that is

$$-A_1(x - \bar{x}) + (B_1 + C_1)(\tilde{f}(y) - \tilde{f}(\bar{y})) = 0,$$
$$-A_2(y - \bar{y}) + (B_2 + C_2)(\tilde{g}(x) - \tilde{g}(\bar{x})) = 0.$$

It is easy to see that

$$2[(x - \bar{x})^T(P_{11} + R_{13} + P_{12}^T) + (\tilde{g}^T(x) - \tilde{g}^T(\bar{x}))R_{23}] \times\{-A_1(x - \bar{x}) + (B_1 + C_1)(\tilde{f}(y) - \tilde{f}(\bar{y}))\} = 0, \quad (8)$$

$$2[(y - \bar{y})^T(Q_{11} + S_{13} + Q_{12}^T) + (\tilde{f}^T(y) - \tilde{f}^T(\bar{y}))S_{23}] \times\{-A_2(y - \bar{y}) + (B_2 + C_2)(\tilde{g}(x) - \tilde{g}(\bar{x}))\} = 0. \quad (9)$$

By inequalities (2) and (3), we get

$$-2(y - \bar{y})^T L_1(T_1 + T_2)L_2(y - \bar{y})$$
$$-2(\tilde{f}(y) - \tilde{f}(\bar{y}))^T(T_1 + T_2)(\tilde{f}(y) - \tilde{f}(\bar{y}))$$
$$+2(y - \bar{y})^T(T_1 + T_2)(L_1 + L_2)(\tilde{f}(y) - \tilde{f}(\bar{y})) \ge 0,$$
$$-2(x - \bar{x})^T L_3(T_3 + T_4)L_4(x - \bar{x})$$
$$-2(\tilde{g}(x) - \tilde{g}(\bar{x}))^T(T_3 + T_4)(\tilde{g}(x) - \tilde{g}(\bar{x}))$$
$$+2(x - \bar{x})^T(T_3 + T_4)(L_3 + L_4)(\tilde{g}(x) - \tilde{g}(\bar{x})) \ge 0.$$

These together with Eqs.(8),(9) give

$$\vartheta^T G \vartheta \geq 0, \qquad (10)$$

where

$$\begin{aligned}
G &= \left[G_{ij}\right]_{4\times4},\\
\vartheta^T &= [(x-\bar{x})^T, (\tilde{g}(x)-\tilde{g}(\bar{x}))^T, (y-\bar{y})^T, (\tilde{f}(y)-\tilde{f}(\bar{y}))^T],\\
G_{11} &= -(P_{11}+R_{13}+P_{12}^T)A_1 - A_1(P_{11}+R_{13}^T+P_{12})\\
&\quad -2L_3(T_3+T_4)L_4,\\
G_{12} &= -A_1R_{23}^T+(T_3+T_4)(L_3+L_4),\\
G_{14} &= (P_{11}+R_{13}+P_{12}^T)(B_1+C_1),\\
G_{22} &= -2(T_3+T_4),\quad G_{23}=(B_2+C_2)^T(F_4^T+Q_{12}),\\
G_{24} &= R_{23}(B_1+C_1)+(B_2+C_2)^T S_{23}^T,\\
G_{33} &= -(Q_{11}+S_{13}+Q_{12}^T)A_2 - A_2(Q_{11}+S_{13}^T+Q_{12})\\
&\quad -2L_1(T_1+T_2)L_2,\\
G_{34} &= -A_2S_{23}^T+(T_1+T_2)(L_1+L_2), \backslash G_{44}=-2(T_1+T_2).
\end{aligned}$$

On the other hand, one can infer from inequality (7) (i=j=1) that $\Omega<0$. Let

$$\mathrm{B}=\begin{bmatrix} I & I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & I & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 \end{bmatrix},$$

multiplying Ω by B and B^T on its left and right side respectively, we obtain

$$G+2k\begin{bmatrix}\mathrm{I} & 0\\ 0 & \mathrm{I}\end{bmatrix}\begin{bmatrix}P & 0\\ 0 & Q\end{bmatrix}\begin{bmatrix}\mathrm{I} & 0\\ 0 & \mathrm{I}\end{bmatrix}^T$$
$$+\mathrm{diag}\left\{\left(1-(1-\eta_2)e^{-2k\bar{\sigma}}\right)R, \left(1-(1-\eta_1)e^{-2k\bar{\tau}}\right)S\right\}<0,$$

where $\mathrm{I}=\begin{bmatrix} I & I\\ 0 & 0\end{bmatrix}$.

Note that $k\geq0, P>0, Q>0, R\geq0, S\geq0, 0\leq\eta_i<1$ $(i=1,2)$, $\bar{\tau}>0, \bar{\sigma}>0$, thus $G<0$. Obviously, this contradicts with (10). The contradiction implies that $\mathrm{H}(w)\neq\mathrm{H}(\bar{w})$. Hence, map H is injective.

Next, we show that $\|\mathrm{H}(w)\|\to\infty$ as $\|w\|\to\infty$. To prove this, it suffices to show that $\|\mathrm{H}(w)-\mathrm{H}(0)\|\to\infty$ as $\|w\|\to\infty$. Similar to above proof, from Lemma 1 and assumptions (2),(3) we obtain

$$\begin{aligned}
&[x^T(P_{11}+R_{13}+P_{12}^T)+(\tilde{g}(x)-\tilde{g}(0))^T R_{23}]\\
&\times(\mathrm{H}_1(w)-\mathrm{H}_1(0))+[y^T(Q_{11}+S_{13}+Q_{12}^T)\\
&+(\tilde{f}(y)-\tilde{f}(0))^T S_{23}]\times(\mathrm{H}_2(w)-\mathrm{H}_2(0))\\
&\leq\frac{1}{2}\begin{bmatrix}x\\ \tilde{g}(x)-\tilde{g}(0)\\ y\\ \tilde{f}(y)-\tilde{f}(0)\end{bmatrix}^T G\begin{bmatrix}x\\ \tilde{g}(x)-\tilde{g}(0)\\ y\\ \tilde{f}(y)-\tilde{f}(0)\end{bmatrix}\\
&\leq\frac{1}{2}\lambda_M(G)\begin{bmatrix}x\\ \tilde{g}(x)-\tilde{g}(0)\\ y\\ \tilde{f}(y)-\tilde{f}(0)\end{bmatrix}^T\begin{bmatrix}x\\ \tilde{g}(x)-\tilde{g}(0)\\ y\\ \tilde{f}(y)-\tilde{f}(0)\end{bmatrix}\\
&\leq2\lambda_M(G)\left\{x^Tx+(\tilde{g}(x)-\tilde{g}(0))^T(\tilde{g}(x)-\tilde{g}(0))\right.\\
&\qquad\left.+y^Ty+(\tilde{f}(y)-\tilde{f}(0))^T(\tilde{f}(y)-\tilde{f}(0))\right\}\\
&\leq2\lambda_M(G)\left(x^Tx+x^T\Upsilon_2^2x+y^Ty+y^T\Upsilon_1^2y\right)\\
&\leq2\lambda_M(G)\left\{(1+\lambda_M(\Upsilon_2))\|x\|^2+(1+\lambda_M(\Upsilon_1))\|y\|^2\right\}\\
&\leq2\lambda_M(G)(2+\lambda_M(\Upsilon_2)+\lambda_M(\Upsilon_1))\|w\|^2,
\end{aligned}$$

where

$$\Upsilon_1=\mathrm{diag}\{\rho_1,\ldots,\rho_n\}, \rho_i=\max\{|l_{1i}|,|l_{2i}|\},$$
$$\Upsilon_2=\mathrm{diag}\{\tilde{n}_1,\ldots,\tilde{n}_n\}, \tilde{n}_i=\max\{|l_{3i}|,|l_{4i}|\}, i=1,\ldots,n.$$

From assumptions (2),(3) and above discussion we have

$$\lambda_M(\Upsilon_1)>0, \lambda_M(\Upsilon_2)>0, \lambda_M(G)<0.$$

By Schwarz inequality and assumptions (2),(3), we have

$$\begin{aligned}
&-2\lambda_M(G)(2+\lambda_M(\Upsilon_2)+\lambda_M(\Upsilon_1))\|w\|^2\\
&\leq\|[x^T(P_{11}+R_{13}+P_{12}^T)+(\tilde{g}(x)-\tilde{g}(0))^T R_{23}]\\
&\quad\times(\mathrm{H}_1(w)-\mathrm{H}_1(0))+[y^T(Q_{11}+S_{13}+Q_{12}^T)\\
&\quad+(\tilde{f}(y)-\tilde{f}(0))^T S_{23}](\mathrm{H}_2(w)-\mathrm{H}_2(0))\|\\
&\leq(\|x\|\times\|P_{11}+R_{13}+P_{12}^T\|+\|\tilde{g}(x)-\tilde{g}(0)\|\times\|R_{23}\|)\\
&\quad\times\|\mathrm{H}_1(w)-\mathrm{H}_1(0)\|+(\|y\|\times\|Q_{11}+S_{13}+Q_{12}^T\|\\
&\quad+\|\tilde{f}(y)-\tilde{f}(0)\|\times\|S_{23}\|)\|\mathrm{H}_2(w)-\mathrm{H}_2(0)\|\\
&\leq(\|P_{11}+R_{13}+P_{12}^T\|+\|\Upsilon_2\|\times\|R_{23}\|)\|x\|\\
&\quad\times\|\mathrm{H}_1(w)-\mathrm{H}_1(0)\|+(\|Q_{11}+S_{13}+Q_{12}^T\|\\
&\quad+\|\Upsilon_1\|\times\|S_{23}\|)\|y\|\times\|\mathrm{H}_2(w)-\mathrm{H}_2(0)\|\\
&\leq\Pi\|\mathrm{H}(w)-\mathrm{H}(0)\|,
\end{aligned}$$

where $\Pi=\|P_{11}+R_{13}+P_{12}^T\|+\|\Upsilon_2\|\times\|R_{23}\|+\|Q_{11}+S_{13}+Q_{12}^T\|+\|\Upsilon_1\|\times\|S_{23}\|$. That is

$$\|\mathrm{H}(w)-\mathrm{H}(0)\|\geq-2\lambda_M(G)\left(2+\sum_{i=1}^{2}\lambda_M(\Upsilon_i)\right)\frac{\|w\|}{\Pi},$$

4. Exponential stability result of the equilibrium point

In order to prove the robust stability of the equilibrium point (x^*, y^*) of neural network (1), we will first simplify neural network (1) as follows. Let $u(t)=x(t)-x^*$, $v(t)=y(t)-y^*$, then we have

$$\begin{cases}\dot{u}(t)= & -\bar{A}_1u(t)+\bar{B}_1f(v(t))+\bar{C}_1f(v(t-\tau(t)))\\ & +\bar{D}_1\dot{u}(t-\sigma(t)),\\ \dot{v}(t)= & -\bar{A}_2v(t)+\bar{B}_2g(u(t))+\bar{C}_2g(u(t-\sigma(t)))\\ & +\bar{D}_2\dot{v}(t-\tau(t)),\end{cases} \qquad (11)$$

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where $f_i(v_i(t)) = \tilde{f}_i(v_i(t)+y_i^*) - \tilde{f}_i(y_i^*), g_j(u_j(t)) = \tilde{g}_j(u_j(t)+x_j^*) - \tilde{g}_j(x_j^*)$ with $f_i(0)=g_j(0)=0, i=1,2, \ldots, m;\ j=1,2,\ldots,n$. By assumptions (2) and (3), we can see that

$$l_{1i} \le \frac{f_i(s_1)-f_i(s_2)}{s_1-s_2} \le l_{2i}, \tag{12}$$

$$l_{3j} \le \frac{g_j(s_1)-g_j(s_2)}{s_1-s_2} \le l_{4j}. \tag{13}$$

Next, we consider neural network (11) without uncertainties, that is

$$\begin{cases} \dot{u}(t) = & -A_1u(t) + B_1 f(v(t)) + C_1 f(v(t-\tau(t))) \\ & +D_1\dot{u}(t-\sigma(t)), \\ \dot{v}(t) = & -A_2v(t) + B_2 g(u(t)) + C_2 g(u(t-\sigma(t))) \\ & +D_2\dot{v}(t-\tau(t)). \end{cases} \tag{14}$$

Theorem 2. The unique equilibrium point of neural network (14) is stable with exponential convergence rate k if the conditions of Theorem 1 are satisfied.

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(u(t)) = \sum_{i=1}^{3} V_i(u(t)) \tag{15}$$

with

$$\begin{aligned} V_1(u(t)) = & \ e^{2kt}\alpha^T(t)P\alpha(t) + e^{2kt}\beta^T(t)Q\beta(t), \\ V_2(u(t)) = & \int_{t-\sigma(t)}^{t} e^{2ks}\gamma^T(s)R\gamma(s)\mathrm{d}s \\ & +\int_{t-\tau(t)}^{t} e^{2ks}\delta^T(s)S\delta(s)\mathrm{d}s, \\ V_3(u(t)) = & \ \bar{\sigma}\int_{t-\bar{\sigma}}^{t}(s-t+\bar{\sigma})e^{2ks}\dot{u}^T(s)U_1\dot{u}(s)\mathrm{d}s \\ & +\bar{\tau}\int_{t-\bar{\tau}}^{t}(s-t+\bar{\tau})e^{2ks}\dot{v}^T(s)U_2\dot{v}(s)\mathrm{d}s, \end{aligned}$$

where $\alpha^T(t) = [u^T(t), u^T(t-\sigma(t))]$, $\beta^T(t) = [v^T(t), v^T(t-\tau(t))]$, $\gamma^T(s) = [u^T(s), g^T(u(s)), \dot{u}^T(s)]$, $\delta^T(s) = [v^T(s), f^T(v(s)), \dot{v}^T(s)]$.

For convenience, we denote $u_\sigma = u(t-\sigma(t)),\ v_\tau = v(t-\tau(t))$. The time derivative of functional (15) along the trajectories of neural network (14) is obtained as follows:

$$\begin{aligned} \dot{V}_1(u(t)) = e^{2kt}\{ & \ 2k\alpha^T(t)P\alpha(t) + 2\alpha^T(t)P\dot{\alpha}(t) \\ & +2k\beta^T(t)Q\beta(t) + 2\beta^T(t)Q\dot{\beta}(t)\}, \\ \dot{V}_2(u(t)) = e^{2kt}\{ & \ \gamma^T(t)R\gamma(t) + \delta^T(t)S\delta(t) \\ & -(1-\dot{\sigma}(t))e^{-2k\sigma(t)}\gamma^T(t-\sigma(t))R\gamma(t-\sigma(t)) \\ & -(1-\dot{\tau}(t))e^{-2k\tau(t)}\delta^T(t-\tau(t))S\delta(t-\tau(t))\}, \end{aligned}$$

$$\begin{aligned} \dot{V}_3(u(t)) = e^{2kt}\{ & \ \bar{\sigma}^2\dot{u}^T(t)U_1\dot{u}(t) + \bar{\tau}^2\dot{v}^T(t)U_2\dot{v}(t) \\ & -\bar{\sigma}\int_{t-\bar{\sigma}}^{t} e^{2k(s-t)}\dot{u}^T(s)U_1\dot{u}(s)\mathrm{d}s \\ & -\bar{\tau}\int_{t-\bar{\tau}}^{t} e^{2k(s-t)}\dot{v}^T(s)U_2\dot{v}(s)\mathrm{d}s\}. \end{aligned}$$

Based on Leibniz-Newton formula, for any real matrix $X_i (i=1,\ldots,4)$ with compatible dimensions, we get

$$\begin{aligned} 0 = 2e^{2kt} & \{u^T(t)X_1^T + u_\sigma^T X_2^T\} \\ & \times\{u(t) - u_\sigma - \int_{t-\sigma(t)}^{t}\dot{u}(s)\mathrm{d}s\}, \end{aligned} \tag{16}$$

$$\begin{aligned} 0 = 2e^{2kt} & \{u_\sigma^T X_3^T + u^T(t-\bar{\sigma})X_4^T\} \\ & \times\{u_\sigma - u(t-\bar{\sigma}) - \int_{t-\bar{\sigma}}^{t-\sigma(t)}\dot{u}(s)\mathrm{d}s\}, \end{aligned} \tag{17}$$

$$\begin{aligned} 0 = 2e^{2kt} & \{v^T(t)Y_1^T + v_\tau^T Y_2^T\} \\ & \times\{v(t) - v_\tau - \int_{t-\tau(t)}^{t}\dot{v}(s)\mathrm{d}s\}, \end{aligned} \tag{18}$$

$$\begin{aligned} 0 = 2e^{2kt} & \{v_\tau^T Y_3^T + v^T(t-\bar{\tau})Y_4^T\} \\ & \times\{v_\tau - v(t-\bar{\tau}) - \int_{t-\bar{\tau}}^{t-\tau(t)}\dot{v}(s)\mathrm{d}s\}. \end{aligned} \tag{19}$$

It is easy to get the following inequalities by using Lemma 1:

$$\begin{aligned} -2\{ & u^T(t)X_1^T + u_\sigma^T X_2^T\}\int_{t-\sigma(t)}^{t}\dot{u}(s)\mathrm{d}s \\ \le & \ \bar{\sigma}e^{-2k\bar{\sigma}}\int_{t-\sigma(t)}^{t}\dot{u}^T(s)U_1\dot{u}(s)\mathrm{d}s \\ & +\frac{1}{\bar{\sigma}}e^{2k\bar{\sigma}}\sigma(t)\zeta^T(t)\Phi_1^T U_1^{-1}\Phi_1\zeta(t), \end{aligned} \tag{20}$$

$$\begin{aligned} -2\{ & u_\sigma^T X_3^T + u^T(t-\bar{\sigma})X_4^T\}\int_{t-\bar{\sigma}}^{t-\sigma(t)}\dot{u}(s)\mathrm{d}s \\ \le & \ \bar{\sigma}e^{-2k\bar{\sigma}}\int_{t-\bar{\sigma}}^{t-\sigma(t)}\dot{u}^T(s)U_1\dot{u}(s)\mathrm{d}s \\ & +\frac{1}{\bar{\sigma}}e^{2k\bar{\sigma}}(\bar{\sigma}-\sigma(t))\zeta^T(t)\Phi_2^T U_1^{-1}\Phi_2\zeta(t), \end{aligned} \tag{21}$$

$$\begin{aligned} -2\{ & v^T(t)Y_1^T + v_\tau^T Y_2^T\}\int_{t-\tau(t)}^{t}\dot{v}(s)\mathrm{d}s \\ \le & \ \bar{\tau}e^{-2k\bar{\tau}}\int_{t-\tau(t)}^{t}\dot{v}^T(s)U_2\dot{v}(s)\mathrm{d}s \\ & +\frac{1}{\bar{\tau}}e^{2k\bar{\tau}}\tau(t)\zeta^T(t)\Psi_1^T U_2^{-1}\Psi_1\zeta(t), \end{aligned} \tag{22}$$

$$\begin{aligned} -2\{ & v_\tau^T Y_3^T + v^T(t-\bar{\tau})Y_4^T\}\int_{t-\bar{\tau}}^{t-\tau(t)}\dot{v}(s)\mathrm{d}s \\ \le & \ \bar{\tau}e^{-2k\bar{\tau}}\int_{t-\bar{\tau}}^{t-\tau(t)}\dot{v}^T(s)U_2\dot{v}(s)\mathrm{d}s \\ & +\frac{1}{\bar{\tau}}e^{2k\bar{\tau}}(\bar{\tau}-\tau(t))\zeta^T(t)\Psi_2^T U_2^{-1}\Psi_2\zeta(t), \end{aligned} \tag{23}$$

where

$$\begin{aligned} \zeta^T(t) = [& \ u^T(t), u_\sigma^T, u^T(t-\bar{\sigma}), g^T(u(t)), g^T(u_\sigma), \\ & \dot{u}^T(t-\sigma(t)), \dot{u}^T(t), v^T(t), v_\tau^T, v^T(t-\bar{\tau}), \\ & f^T(v(t)), f^T(v_\tau), \dot{v}^T(t-\tau(t)), \dot{v}^T(t)\ ]. \end{aligned}$$

On the other hand, one can infer from inequalities (12),(13) that the following matrix inequalities hold for any positive diagonal matrices $T_i (i = 1,2,3,4)$ with compatible dimensions

$$0 \le -2e^{2kt}\{ v^T(t)L_2T_1L_1v(t) + f^T(v(t))T_1f(v(t)) - v^T(t)T_1(L_1+L_2)f(v(t))\}, \quad (24)$$

$$0 \le -2e^{2kt}\{ v_\tau^T L_2T_2L_1v_\tau + f^T(v_\tau)T_2f(v_\tau) - v_\tau^T T_2(L_1+L_2)f(v_\tau)\}, \quad (25)$$

$$0 \le -2e^{2kt}\{ u^T(t)L_4T_3L_3u(t) + g^T(u(t))T_3g(u(t)) - u^T(t)T_3(L_3+L_4)g(u(t))\}, \quad (26)$$

$$0 \le -2e^{2kt}\{ u_\sigma^T L_4T_4L_3u_\sigma + g^T(u_\sigma)T_4g(u_\sigma) - u_\sigma^T T_4(L_3+L_4)g(u_\sigma)\}. \quad (27)$$

To get less conservative criterion, we introduce the following equalities for any real matrices Z_1, Z_2 with compatible dimensions

$$0 = 2\dot{u}^T(t)Z_1\{ -\dot{u}(t) - A_1u(t) + B_1f(v(t)) + C_1f(v_\tau) + D_1\dot{u}^T(t-\sigma(t))\}, \quad (28)$$

$$0 = 2\dot{v}^T(t)Z_2\{ -\dot{v}(t) - A_2v(t) + B_2g(u(t)) + C_2g(u_\sigma) + D_2\dot{v}^T(t-\tau(t))\}. \quad (29)$$

From (15)-(29), we obtain

$$\dot{V}(u(t)) \le e^{2kt}\zeta^T(t) \Big(\Omega + \frac{1}{\bar\sigma}e^{2k\bar\sigma}\sigma(t)\Phi_1^TU_1^{-1}\Phi_1 + \frac{1}{\bar\sigma}e^{2k\bar\sigma}(\bar\sigma-\sigma(t))\Phi_2^TU_1^{-1}\Phi_2 + \frac{1}{\bar\tau}e^{2k\bar\tau}\tau(t)\Psi_1^TU_2^{-1}\Psi_1 + \frac{1}{\bar\tau}e^{2k\bar\tau}(\bar\tau-\tau(t))\Psi_2^TU_2^{-1}\Psi_2\Big)\zeta(t).$$

Note that $0 \le \tau(t) \le \bar\tau$, $0 \le \sigma(t) \le \bar\sigma$, so

$$\Omega + \frac{1}{\bar\sigma}e^{2k\bar\sigma}\sigma(t)\Phi_1^TU_1^{-1}\Phi_1 + \frac{1}{\bar\tau}e^{2k\bar\tau}\tau(t)\Psi_1^TU_2^{-1}\Psi_1 + \frac{1}{\bar\sigma}e^{2k\bar\sigma}(\bar\sigma-\sigma(t))\Phi_2^TU_1^{-1}\Phi_2 + \frac{1}{\bar\tau}e^{2k\bar\tau}(\bar\tau-\tau(t))\Psi_2^TU_2^{-1}\Psi_2 < 0$$

holds if and only if the following four inequalities

$$\Omega + e^{2k\bar\sigma}\Phi_1^TU_1^{-1}\Phi_1 + e^{2k\bar\tau}\Psi_1^TU_2^{-1}\Psi_1 < 0, \quad (30)$$

$$\Omega + e^{2k\bar\sigma}\Phi_1^TU_1^{-1}\Phi_1 + e^{2k\bar\tau}\Psi_2^TU_2^{-1}\Psi_2 < 0, \quad (31)$$

$$\Omega + e^{2k\bar\sigma}\Phi_2^TU_1^{-1}\Phi_2 + e^{2k\bar\tau}\Psi_1^TU_2^{-1}\Psi_1 < 0, \quad (32)$$

$$\Omega + e^{2k\bar\sigma}\Phi_2^TU_1^{-1}\Phi_2 + e^{2k\bar\tau}\Psi_2^TU_2^{-1}\Psi_2 < 0, \quad (33)$$

are true. From the well-known Schur complement, inequalities (30-33) are equivalent to (7) with *i,j=1,2* respectively, thus $\dot{V}(u(t)) < 0$ holds if (7) (*i,j=1,2*) are true.

Furthermore, following the similar line in [27], from Lemma 1 we have

$$V(u(0)) \le M_1\|\phi(t)-x^*\|^2 + M_2 \sup_{-\bar\sigma\le\theta\le0}\|\dot{u}(\theta)\|^2 + M_3\|\varphi(t)-y^*\|^2 + M_4 \sup_{-\bar\tau\le\theta\le0}\|\dot{v}(\theta)\|^2,$$

where

$$M_1 = 4\lambda_M(P) + 3\bar\sigma\lambda_M(R)\big(1+\sigma_1^2\big),$$

$$M_2 = 3\bar\sigma\lambda_M(R) + \frac{1}{2}\bar\sigma^3\lambda_M(U_1),$$

$$M_3 = 4\lambda_M(Q) + 3\bar\tau\lambda_M(S)\big(1+\sigma_2^2\big),$$

$$M_4 = 3\bar\tau\lambda_M(S) + \frac{1}{2}\bar\tau^3\lambda_M(U_2),$$

and $\sigma_1 = \max_{1\le i\le n}\{|l_{3i}|, |l_{4i}|\}$, $\sigma_2 = \max_{1\le i\le n}\{|l_{1i}|, |l_{2i}|\}$.

Meanwhile

$$V(u(t)) \ge e^{2kt}\big(\lambda_m(P_{11})\|\phi(t)-x^*\|^2 + \lambda_m(Q_{11}) \times \|\varphi(t)-y^*\|^2\big) \ge \frac{1}{2}e^{2kt} \min\{\lambda_m(P_{11}), \lambda_m(Q_{11})\} \times\big(\|\phi(t)-x^*\| + \|\varphi(t)-y^*\|\big)^2,$$

by Lyapunov stability theory, the proof of Theorem 1 is completed.

Remark 1. One can notice that the augmented Lyapunov functional approach of this paper is quite different from previous ones. New terms $e^{2kt}u^T(t-\sigma(t))P_{22}u(t-\sigma(t))$ and $e^{2kt}v^T(t-\tau(t))Q_{22}v(t-\tau(t))$ are used to augment the Lyapunov functional, whose derivatives are directly coupled with both neutral and retarded systems. Therefore, the augmented Lyapunov functional can lead to an reduce in the conservativeness of the results, which will be illustrated by three examples.

Remark 2. The proposed Lyapunov functional also includes the terms of cross products $e^{2kt}u^T(t)P_{12}u(t-\sigma(t))$, $e^{2kt}v^T(t)Q_{12}v(t-\tau(t))$, and some integral terms of cross products, such as $\int_{t-\sigma(t)}^{t} e^{2ks}u^T(s)R_{12}g(u(s))\mathrm{d}s$, $\int_{t-\tau(t)}^{t} e^{2ks}v^T(s)S_{12}f(v(s))\mathrm{d}s$, which are not considered in previous results. This approach provides a quasi-full-size Lyapunov functional through augmentation. Moreover, the set of Lyapunov-Krasovskii functional introduced in Park et al. [14] is just reduced form of the one proposed in this paper. It is well known that as the Lyapunov functional is reduced, the corresponding results become more conservative. Therefore the proposed novel augmented Lyapunov functional can yield less conservative results than the existing methods.

Remark 3. It should be pointed out that the condition of Theorem 1 in [14] needs to be revised. In the proof, one key step is the following proposition (see inequality (14) in Page 719 of [14]):

Proposition. Let $L_3=-L_4$. The following inequality holds for any given $L_4>0$, positive definite matrices Z:

$$g^T(y(t))Zg(y(t))\le y^T(t)L_4ZL_4y(t). \qquad (34)$$

Unfortunately, the above Proposition is not valid in general, this fact can be illustrated by the following example:

Example 3.1. Obviously $g_1(s)=g_2(s)=\frac{1}{2}(|s+1|-|s-1|)$ satisfy conditions (3) with $L_4=I$. Set $y^T(t)=[-2\ 1]$, then $g^T(y(t))=[-1\,1]$. Further set $Z=\begin{bmatrix}1&2\\2&5\end{bmatrix}$, we have

$$g^T(y(t))Zg(y(t))=2>1=y^T(t)L_4ZL_4y(t),$$

i.e. inequality (34) is false.

Based on above analysis, the LMIs of Theorem 1 in [14] may not be sufficient conditions assumed that Z is a positive definite matrix. In fact, if Z is revised to be a positive scalar matrix, then the above Proposition and Theorem 1 in [14] are still valid.

Remark 4. If any of $\dot{\sigma}(t),\dot{\tau}(t)$ are unknown or any of $\sigma(t),\tau(t)$ are not differentiable, by setting $P_{i2}=Q_{i2}=0$ $(i=1,2), R=S=0$ in functional (15) and adding the following zero equations to the time derivative along the trajectories of neural network (14):

$$\begin{aligned}0=\dot{u}^T(t-\sigma(t))K_1\big(&-\dot{u}(t)-A_1u(t)+B_1f(v(t))\\&+C_1f(v_\tau))+D_1\dot{u}(t-\sigma(t))\big),\\0=\dot{v}^T(t-\tau(t))K_2\big(&-\dot{v}(t)-A_2v(t)+B_2g(u(t))\\&+C_2g(u_\sigma)+D_2\dot{v}(t-\tau(t))\big),\end{aligned}$$

where $K_i(i=1,2)$ are any real matrices with compatible dimensions. Following the similar line in Theorem 1 we can obtain a stability criterion similar to LMIs (7).

Remark 5. In terms of LMIs, Theorem 1 and Remark 4 provide a sufficient condition for the global exponential stability of the delayed neutral-type neural network in (14). One of the advantages of the LMI approach is that the LMI condition can be checked numerically very efficiently by using the interior-point algorithms, which have been developed recently in solving LMIs [1].

Next, consider system (14) with $D_1=D_2=0$, that is

$$\begin{cases}\dot{u}(t)=-A_1u(t)+B_1f(v(t))+C_1f(v_\tau),\\\dot{v}(t)=-A_2v(t)+B_2g(u(t))+C_2g(u_\sigma).\end{cases} \qquad (35)$$

It is easy to see the following result holds from Theorem 1.

Corollary 1. Under assumptions (2),(3) and $0\le\tau(t)\le\bar{\tau}$, $0\le\sigma(t)\le\bar{\sigma}, 0\le\dot{\tau}(t)\le\eta_1<1, 0\le\dot{\sigma}(t)\le\eta_2<1$, given a constant $k\ge0$, suppose that there exist positive definite symmetric matrices P,Q, nonnegative definite symmetric matrices $R,S,Z_i,U_i(i=1,2)$, positive diagonal matrices T_j, real matrices $X_j,Y_j(j=1,2,3,4)$ with compatible dimensions such that the LMIs (7) (*i*,*j*=1,2) with $D_1=D_2=0$ hold, then the equilibrium point of neural network (35) is exponential stable.

Remark 6. In Corollary 1, by setting $P_{i2}=Q_{i2}=0(i=1,2), R=S=0,$ we can employ this criterion to analyze the stability of neural network (35) when any of $\dot{\sigma}(t),\dot{\tau}(t)$ are unknown or any of $\sigma(t),\tau(t)$ are not differentiable.

5. Robust exponential stability results of uncertain delayed neural network

Now, based on Lemma 1 we investigate the robust exponential stability condition for neural network (11) with uncertainties satisfying (4) and (5). Firstly, by using the same functional as in Theorem 1, we can easily obtain the following result.

Theorem 3. Under assumptions (2)-(5) and $0\le\tau(t)\le\bar{\tau}$, $0\le\sigma(t)\le\bar{\sigma}, 0\le\dot{\tau}(t)\le\eta_1<1, 0\le\dot{\sigma}(t)\le\eta_2<1$, given a constant $k\ge0$, suppose that there exist positive scalars $\varepsilon_0,\varepsilon_1$, positive definite symmetric matrices P,Q, nonnegative definite symmetric matrices $R,S,Z_i,U_i(i=1,2)$, positive diagonal matrices T_j, real matrices $X_j,Y_j(j=1,2,3,4)$ with compatible dimensions such that the following LMIs hold (*i*,*j*=1,2):

$$\begin{bmatrix}\Omega+\Theta & \Xi_0^TH_0 & \Xi_1^TH_1 & \Phi_i^T & \Psi_j^T\\ H_0^T\Xi_0 & -\varepsilon_0I & 0 & 0 & 0\\ H_1^T\Xi_1 & 0 & -\varepsilon_1I & 0 & 0\\ \Phi_i & 0 & 0 & -e^{-2k\bar{\sigma}}U_1 & 0\\ \Psi_j & 0 & 0 & 0 & -e^{-2k\bar{\tau}}U_2\end{bmatrix}<0, \qquad (36)$$

where

$$\begin{aligned}\Theta=\ &\varepsilon_0[-G_1\ 0\ 0\ 0\backslash 0\ G_4\ 0\ 0\ 0\ 0\ G_2\ G_3\ 0\ 0]^T\\&\times[-G_1\ 0\ 0\ 0\backslash 0\ G_4\ 0\ 0\ 0\ 0\ G_2\ G_3\ 0\ 0]\end{aligned}$$

$$+\varepsilon_1[0\ 0\ 0\ G_6\ G_7\ 0\ 0\ -G_5 \backslash 0\ 0 \backslash 0\ 0\ G_8\ 0]^T$$
$$\times[0\ 0\ 0\ G_6\ G_7\ 0\ 0\ -G_5 \backslash 0\ 0\ 0\ 0\ G_8\ 0],$$
$$\Xi_0 = [P_{11}+R_{13}^T\ P_{12}\ 0\ R_{23}^T\ 0 \backslash 0\ Z_1^T\ 0\ 0\ 0\ 0\ 0\ 0\ 0],$$
$$\Xi_1 = [0\ 0\ 0\ 0\ 0\ 0\ 0\ Q_{11}+S_{13}^T \backslash Q_{12}\ 0\ S_{23}^T\ 0\ 0\ Z_2^T],$$

and other parameters are all defined in Theorem 1, then the equilibrium point of neural network (1) is robust exponential stable

Remark 7. Similar to Remark 4, if any of $\sigma(t),\tau(t)$ are not differentiable or any of $\dot\sigma(t),\dot\tau(t)$ are unknown, by setting $P_{i2}=Q_{i2}=0(i=1,2), R=S=0$ in functional (15) and adding the following zero equations

$$0=\dot u^T(t-\sigma(t))K_1\Big(-\dot u(t)-\bar A_1 u(t)+\bar B_1 f(v(t)) +\bar C_1 f(v_\tau))+\bar D_1\dot u(t-\sigma(t))\Big),$$
$$0=\dot v^T(t-\tau(t))K_2\Big(-\dot v(t)-\bar A_2 v(t)+\bar B_2 g(u(t)) +\bar C_2 g(u_\sigma)+\bar D_2\dot v(t-\tau(t))\Big),$$

to the time derivative along the trajectories of neural network (11), following the similar line in Theorem 3 we can obtain a stability criterion similar to LMIs (36).

Now, consider system (11) with $D_1=D_2=0$, that is

$$\dot u(t) = -\bar A_1 u(t)+\bar B_1 f(v(t))+\bar C_1 f(v(t-\tau(t))),$$
$$\dot v(t) = -\bar A_2 v(t)+\bar B_2 g(u(t))+\bar C_2 g(u(t-\sigma(t))).$$

It is easy to see the following result holds from Theorem 3.

Corollary 2. Under assumptions (2),(3) and $0\le\tau(t)\le\bar\tau$, $0\le\sigma(t)\le\bar\sigma, 0\le\dot\tau(t)\le\eta_1<1, 0\le\dot\sigma(t)\le\eta_2<1$, given a constant $k\ge 0$, suppose that there exist positive scalars $\varepsilon_0,\varepsilon_1$, positive definite symmetric matrices P,Q, nonnegative definite symmetric matrices $R,S,U_i(i=1,2)$, positive diagonal matrices T_j, real matrices $X_j,Y_j(j=1,2,3,4)$ with compatible dimensions such that the LMIs (36) with $D_1=D_2=0$ hold, then the equilibrium point of neural network (37) is robust exponential stable.

Remark 8. In Corollary 2, by setting $P_{i2}=Q_{i2}=0(i=1,2), R=S=0$, we can employ this criterion to analyze the stability of neural network (37) when any of $\dot\sigma(t),\dot\tau(t)$ are unknown or any of $\sigma(t),\tau(t)$ are not differentiable.

6. Comparison and Illustrative Examples

Next, we provide two numerical examples to demonstrate the effectiveness and less conservativeness of our delay-dependent stability criteria over some recent results in the literature.

Example 5.1. Consider neural network (35) with

$$A_1=I, A_2=4I, B_1=B_2=0,$$
$$C_1=\begin{bmatrix}0.05 & 0.10 & 0.15\\ 0.25 & 0.05 & 0.15\\ 0.05 & 0.15 & 0.05\end{bmatrix},$$
$$C_2=\begin{bmatrix}0.75 & 0 & 0.15\\ 0.75 & 0.50 & 0.95\\ 0.95 & 0.75 & 0.95\end{bmatrix},$$
$$L_1=L_3=0, L_2=L_4=I.$$

This model was studied in [7], [8]. For this neural network, it is verified in [8] that the results given in [2], [3] fail to ascertain the stability for any time delay. Furthermore, if we set exponential convergence rate k be fixed as 0.35, none of the criteria of [4], [19] can guarantee the stability for any time delay with $\dot\tau(t)\ne 0$ or $\dot\sigma(t)\ne 0$. Set $\tau(t)\equiv 0.5110, \eta_2\ge 1$, all of the criteria given in [8], [17] fail to verify the stability for any time delay, the allowable time delay upper bound obtained by Gau et al. [7] is 0.5110, while our method shows that the equilibrium point of this neural network is exponentially stable for any time delay with $\sigma(t)\le 1.1061$. This is much larger than the one of [7], which shows the less conservativeness of our developed method.

Example 5.2. Consider neural network (6) with

$$A_1=\mathrm{diag}\{2,1.5\},\ A_2=\mathrm{diag}\{3,2\},$$
$$B_1=\begin{bmatrix}0.3 & -1.0\\ 0.1 & 0.2\end{bmatrix}, B_2=\begin{bmatrix}0.2 & 0.5\\ 1.1 & 0.2\end{bmatrix},$$
$$C_1=\begin{bmatrix}1.1 & -0.2\\ 1.0 & 0.5\end{bmatrix}, C_2=\begin{bmatrix}c & 0\\ c & c\end{bmatrix}, c>0,$$
$$D_1=\begin{bmatrix}0.2 & 0.1\\ 0.1 & 0.2\end{bmatrix}, D_2=\begin{bmatrix}0.1 & 0.2\\ 0.2 & 0.1\end{bmatrix},$$
$$J=\begin{bmatrix}-6 & 2\end{bmatrix}^T,\ E=\begin{bmatrix}4 & -2\end{bmatrix}^T,$$
$$\tilde f_1(s)=\tilde f_2(s)=\tilde g_1(s)=\tilde g_2(s)=0.5(|s+1|-|s-1|).$$

Obviously, the activation functions satisfy assumptions (2) and (3) with $L_1=L_3=0, L_2=L_4=I$.

Obviously, none of the results in [2]–[4], [7], [8], [12], [15] and [19] can be applied to verify the stability of this model. However, if we set $\bar\sigma=\bar\tau=1$, $\eta_1=\eta_2=0.5$, from Theorems 1 and 2 we can conclude that this neural network has a unique equilibrium point

$(1.1167, -1.4500, -1.7000, 1.6000)^T$ which is exponential stable for any c with $0 < c \le 0.9506$.

7. Conclusion

In this paper we have investigated the global robust stability problem of uncertain BAM neural networks of neutral-type. By employing new Lyapunov-Krasovskii functional, we proposed several novel stability criteria for the considered neural networks. The obtained results are all in the form of LMI, which can be easily optimized. Finally, three examples are given to show the superiority of our proposed stability conditions to some existing ones.

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