Existence and Local Behavior of the Cubic Padé Approximation

Lei Shi [1], and Zhi-Min Xu [2]

[1] School of Software, Dalian Jiaotong University, Dalian, 116028, P. R. China
[2] School of Science, Dalian Jiaotong University, Dalian, 116028, P. R. China

Abstract

This paper analyses the local behavior of the cubic function approximation of the form $P(z)f^3(z)+Q(z)f(z)+R(z)=O(z^{p+q+r+2})$, where $P(z),Q(z),R(z)$ are algebraic polynomials of degree *p,q,r* respectively, to a function which has a given power series expansion about the origin. It is shown that the cubic Hermite-Padé form always defines a cubic function and that this function is analytic in a neighbourhood of the origin.

Keywords: *Cubic function approximation, Hermite-Padé approximation, algebraic polynomials*

1. Introduction

The Padé approximation theory has been widely used in problems of theoretical physics [1,3], numerical analysis [6] [7], and electrical engineering, especially in modal analysis model [2], order reduction of multivariable systems [4,8].

Let *f(z)* be a function, analytic in some neighbourhood of the origin, whose series expansion about the origin is known. In this paper we wish to consider the properties of the cubic Hermite-Padé approximation approximations to *f(z)* generated by finding polynomials *P(z),T(z),Q(z)* and *R(z)* such that

$$P(z)f^3(z)+T(z)f^2(z)+Q(z)f(z)+R(z)=O(z^{p+t+q+r+3}),$$

with *P(z),T(z),Q(z),R(z)* being algebraic polynomials of degree *p,t,q,r* respectively. But as is well known, if we set

$$z=y-\frac{a}{3},$$

then any cubic equation

$$z^3+az^2+bz+c=0$$

can be transformed into the following form

$$y^3+(b-\frac{a^2}{3})y+(\frac{2}{27}a^3-\frac{1}{3}ab+c)=0.$$

So without loss of generality, in this paper we only consider approximations to *f(z)* generated by finding polynomials *P(z),Q(z)* and *R(z)* so that

$$P(z)f^3(z)+Q(z)f(z)+R(z)=O(z^{p+q+r+2}). \quad (1)$$

Note that such polynomials *P,Q,R* not all zero, must exist since (1) represents a homogenous system of *p+q+r+2* linear equations in the *p+q+r+3* unknown coefficients of the *P(z),Q(z),R(z)*. Then set

$$P(z)u^3(z)+Q(z)u(z)+R(z)=0$$

and attempt to solve this equation for *u(z)* in such a way that *u(z)* approximates *f(z)*.

In the well-known case of Padé approximation [1], the same procedure is followed by

$$P(z)f(z)+Q(z)=O(z^{p+q+1})$$

which gives

$$u(z)=-\frac{Q(z)}{P(z)}.$$

If $P(0)\neq 0$ (not a serious restriction), it then follows that

$$u(z)=f(z)+O(z^{p+q+1}).$$

In the case of quadratic Hermite-Padé approximation [5], the procedure is followed by

$$P(z)u^2(z)+Q(z)u(z)+R(z)=O(z^{p+q+r+2})$$

which gives

$$u(z)=\left(-Q(z)\pm\sqrt{B(z)}\right)/\left(2P(z)\right),$$

where

$$B(z)=Q^2(z)-4P(z)R(z).$$

If $B(0)\neq 0$, it then follows that

$$u(z)=f(z)+O(z^{p+q+r+2}).$$

If *B(0)=0*, we set

$$B(z)=z^{2s}g(z), g(0)\neq 0$$

(since Ref. [5] has proved that *B(z)* never has a root of odd multiplicity at the origin). It then follows that

$$u(z)=f(z)+O(z^{p+q+r+2-s}),$$

where *2s<p+q+r+1*.

However, in the cubic case it is not obvious that

$$P(z)u^3(z)+Q(z)u(z)+R(z)=0$$

yields even an analytic approximation to *f(z)*, still less that it defines a function *u(z)* such that

$$u(z)=f(z)+O(z^{p+q+r+2}).$$

The purpose of this paper is to show that an analogue of the Padé and quadratic Hermite-Padé results is in fact true.

2. Notation

It is assumed that

$$P(z)f^3(z)+Q(z)f(z)+R(z)=O(z^{N+2}),$$

where $N\geq p+q+r$ and that

$$|P(0)|+|Q(0)|+|R(0)|\neq 0.$$

Note that if z^s is the maximal common factor of *P(z)*, *Q(z)*,*R(z)*, then

$$\frac{P(z)}{z^s}f^3(z)+\frac{Q(z)}{z^s}f(z)+\frac{R(z)}{z^s}=O(z^{N+2-s})$$

so that this second assumption is not a serious restriction.

The following notation will be used:
(i) An approximation derived from

$$P(z)f^3(z)+Q(z)f(z)+R(z)=O(z^{N+2})$$

will be referred to as a *(p,q,r)* cubic approximation to *f(z)*.
(ii) Let

$$D(z)=\frac{1}{4}P(z)R^2(z)+\frac{1}{27}Q^3(z).$$

(iii) By $\sqrt{D(z)},\sqrt[3]{E(z)}$ we mean the principal square root of *D(z)*,*E(z)* respectively.

3. The Principal Results

The problem divides itself into two cases, the case *D(0)=0* and the case $D(0)\neq 0$.

3.1 The Case $D(0)\neq 0$

Theorem 1. If $D(0)\neq 0$, then there exists a unique function *u(z)*, analytic in a neighbourhood of the origin, satisfying

$$P(z)f^3(z)+Q(z)f(z)+R(z)=0$$

and *u(0)=f(0)*.

Proof. (i) Suppose $P(0)Q(0)\neq 0$. The three possible expressions for *u(z)* in a neighbourhood of the origin are given by

$$u_k(z)=\omega_1^k\sqrt[3]{-\frac{R(z)}{2P(z)}+\sqrt{\frac{D(z)}{P^3(z)}}}-\omega_2^k\sqrt[3]{\frac{R(z)}{2P(z)}+\sqrt{\frac{D(z)}{P^3(z)}}},$$

$k=0,1,2;$
where

$$\omega_1=\frac{-1+\sqrt{3}i}{2},\omega_2=\frac{-1-\sqrt{3}i}{2}.$$

Since $P(0)Q(0)D(0)\neq 0$, these three functions are all analytic in a neighbourhood of the origin. Exactly one of them satisfies *u(0)=f(0)*, because

$$P(0)f^3(0)+Q(0)f(0)+R(0)=0$$

$$\Rightarrow f(0)=\omega_1^k\sqrt[3]{-\frac{R(0)}{2P(0)}+\sqrt{\frac{D(0)}{P^3(0)}}}-\omega_2^k\sqrt[3]{\frac{R(0)}{2P(0)}+\sqrt{\frac{D(0)}{P^3(0)}}},$$

$k=0,1,2.$

(ii) Suppose *Q(0)=0*. Then $P(0)\neq 0$ (since $D(0)\neq 0$). The three possible expressions for *u(z)* in a neighbourhood of the origin are given by

$$u_k(z)=\omega_1^kQ(z)\Big/\left(3P(z)\sqrt[3]{\frac{R(z)}{2P(z)}+\sqrt{\frac{D(z)}{P^3(z)}}}\right)-\omega_2^k\sqrt[3]{\frac{R(z)}{2P(z)}+\sqrt{\frac{D(z)}{P^3(z)}}},$$

$k=0,1,2.$

Since $P(0)D(0)\neq 0$, these three functions are all analytic in a neighbourhood of the origin. Also exactly one of them satisfies *u(0)=f(0)*, because

$$P(0)f^3(0)+Q(0)f(0)+R(0)=0\Rightarrow$$

$$f(0)=\omega_1^kQ(0)\Big/\left(3P(0)\sqrt[3]{\frac{R(0)}{2P(0)}+\sqrt{\frac{D(0)}{P^3(0)}}}\right)-\omega_2^k\sqrt[3]{\frac{R(0)}{2P(0)}+\sqrt{\frac{D(0)}{P^3(0)}}},$$

$k=0,1,2.$

(iii) Suppose

$$Q(0)\neq 0,P(0)=0.$$

Near the origin the three possible expressions $u_k(z)$ (*k=0,1,2*) can be written as

$$u_k(z)=\omega_1^k\sqrt[3]{-\frac{R}{2P}+\sqrt{\left(\frac{Q}{3P}\right)^3}\sqrt{1+\frac{27PR^2}{4Q^3}}}$$
$$-\omega_2^k\sqrt[3]{\frac{R}{2P}+\sqrt{\left(\frac{Q}{3P}\right)^3}\sqrt{1+\frac{27PR^2}{4Q^3}}}. \qquad (2)$$

The right-hand sides of $u_k(z)$ (*k=0,1,2*) are unbounded as $z\rightarrow 0$, so we can exclude these possibilities. Since *P(0)=0*, close to the origin we can apply the binomial theorem to get from $u_0(z)$ the convergent power series (analytic in a neighbourhood of the origin) expression for *u(z)*.

It follows that

$$u(z)=\begin{cases}\sqrt[3]{-\frac{R(z)}{2P(z)}+\sqrt{\frac{D(z)}{P^3(z)}}}-\sqrt[3]{\frac{R(z)}{2P(z)}+\sqrt{\frac{D(z)}{P^3(z)}}}, & z\neq 0\\ -\frac{3R(z)}{Q^2(z)}, & z=0\end{cases}$$

is the only function, analytic in a neighbourhood of the origin, satisfying

$$P(z)u^3(z)+Q(z)u(z)+R(z)=0$$

with *u(0)=f(0)*.

Theorem 2. If $Q(0)D(0)\neq 0$, then there exists a unique function *u(z)*, analytic in a neighbourhood of the origin, satisfying

$$P(z)u^3(z)+Q(z)u(z)+R(z)=0$$

such that

$$u(z)=f(z)+O(z^{N+2}).$$

Proof. Note that

$$\frac{d^j}{dz^j}\left[P(z)u^3(z)+Q(z)u(z)+R(z)\right]\Big|_0=0$$

$$=\frac{d^j}{dz^j}\left[P(z)f^3(z)+Q(z)f(z)+R(z)\right]\Big|_0, \qquad (3)$$

$j\in\{0,1,\dots,N+1\}$.

For $j=1$

$$\left[\frac{\partial}{\partial u}\left(P(z)u^3(z)+Q(z)u(z)+R(z)\right)u'(z)\right.$$
$$\left.+\left(P'(z)u^3(z)+Q'(z)u(z)+R'(z)\right)\right]\Big|_0=0,$$

$$\left[\frac{\partial}{\partial f}\left(P(z)f^3(z)+Q(z)f(z)+R(z)\right)f'(z)\right.$$
$$\left.+\left(P'(z)f^3(z)+Q'(z)f(z)+R'(z)\right)\right]\Big|_0=0.$$

Differentiating again ($j=2$) gives

$$\left[\frac{\partial}{\partial u}\left(P(z)u^3(z)+Q(z)u(z)+R(z)\right)u''(z)\right.$$
$$+\frac{d}{dz}\left(\frac{\partial}{\partial u}\left(P(z)u^3(z)+Q(z)u(z)+R(z)\right)\right)u'(z)$$
$$\left.+\frac{d}{dz}\left(P'(z)u^3(z)+Q'(z)u(z)+R'(z)\right)\right]\Big|_0=0,$$

$$\left[\frac{\partial}{\partial f}\left(P(z)f^3(z)+Q(z)f(z)+R(z)\right)f''(z)\right.$$
$$+\frac{d}{dz}\left(\frac{\partial}{\partial f}\left(P(z)f^3(z)+Q(z)f(z)+R(z)\right)\right)f'(z)$$
$$\left.+\frac{d}{dz}\left(P'(z)f^3(z)+Q'(z)f(z)+R'(z)\right)\right]\Big|_0=0.$$

In general, more compact form we have

$$\left[\frac{\partial}{\partial u}\left(P(z)u^3(z)+Q(z)u(z)+R(z)\right)u^{(j)}(z)+u_j(z)\right]\Big|_0$$
$$=\left[\frac{\partial}{\partial f}\left(P(z)f^3(z)+Q(z)f(z)+R(z)\right)f^{(j)}(z)+v_j(z)\right]\Big|_0=0, \qquad (4)$$

$j\in\{1,2,\dots,N+1\}$.

where

$$u_1(z)=P'(z)u^3(z)+Q'(z)u(z)+R'(z),$$

$$u_{j+1}(z)=\frac{du_j(z)}{dz}+\frac{d}{dz}\left(\frac{\partial}{\partial u}\left(P(z)u^3(z)+Q(z)u(z)+R(z)\right)\right)u^{(j)}(z);$$

$$v_1(z)=P'(z)f^3(z)+Q'(z)f(z)+R'(z),$$

$$v_{j+1}(z)=\frac{dv_j(z)}{dz}+\frac{d}{dz}\left(\frac{\partial}{\partial u}\left(P(z)f^3(z)+Q(z)f(z)+R(z)\right)\right)f^{(j)}(z).$$

Now taking the unique $u(z)$ from Theorem 1, it is easily proved that

$$\frac{\partial}{\partial u}\left[P(z)u^3(z)+Q(z)u(z)+R(z)\right]_0\neq 0,$$

since

$$D(0)=\left[\frac{1}{4}P(z)R^2(z)+\frac{1}{27}Q^3(z)\right]_0\neq 0,$$

and

$$\left[P(z)u^3(z)+Q(z)u(z)+R(z)\right]_0=0.$$

Therefore it is seen that Eq.(4) with $j=1$ gives $u'(0)=f'(0)$, which with $j=2$ gives $u''(0)=f''(0)$.

It follows that

$$u^{(j)}(0)=f^{(j)}(0),\quad j\in\{1,2,\dots,N+1\},$$

i.e.

$$u(z)=f(z)+O(z^{N+2}).$$

3.2 The Case $D(0)=0$.

We now investigate the case $D(0)=0$. This implies that $P(0)\neq 0$ (since if

$$D(0)=\frac{1}{4}P(0)R^2(0)+\frac{1}{27}Q^3(0)=0$$

and $P(0)=0$, then $Q(0)=0$, which with

$$P(0)f^3(0)+Q(0)f(0)+R(0)=0$$

gives $R(0)=0$; this contradicts the assumption that

$$|P(0)|+|Q(0)|+|R(0)|\neq 0).$$

First, it is necessary to treat two special cases:

(i) Suppose $R(z)\equiv 0$.

Then

$$P(z)f^3(z)+Q(z)f(z)=O(z^{N+2})$$
$$\Rightarrow \left(P(z)f^2(z)+Q(z)\right)f(z)=O(z^{N+2})$$

so that

$$f^2(z)=-Q(z)/P(z)+O(z^S),$$

and

$$f(z)=O(z^T),\ \text{ where } S+T=N+2.$$

Choosing

$$u(z)=\begin{cases}\sqrt{-Q(z)/P(z)}, & \text{if } S>2T\\ 0, & \text{otherwise}\end{cases}$$

gives $u(z)$ such that

$$P(z)u^3(z)+Q(z)u(z)+R(z)=0$$

and

$$u(z)=f(z)+O(z^{\max\{S/2,T\}}).$$

Clearly $\max\{S/2,T\}\geq (N+2)/3$.

(ii) Suppose $Q(z)\equiv 0$.

Then

$$P(z)f^3(z)+R(z)=O(z^{N+2})$$
$$\Rightarrow -R(z)/P(z)=f^3(z)+O(z^{N+2})$$

so that

$$u(z)=-\sqrt[3]{R(z)/P(z)}=f(z)+O(z^K),\quad K\geq (N+2)/3.$$

and

$$P(z)u^3(z)+Q(z)u(z)+R(z)=0.$$

Theorem 3. If $D(z)\not\equiv 0$ and $R(z)\not\equiv 0$, then *D(z)* never has a oot of multiplicity greater than *p+q+2r* at the origin.

Proof. Let *M=p+q+2r* and suppose $D(z)=z^{M+1}D_s(z)$, where $D_s(z)$ is a polynomial of degree *s*. Since $P(z)\not\equiv 0,\ R(z)\not\equiv 0,$ then

$$Q^3(z)=27z^{M+1}D_s(z)+B_t(z), \qquad (5)$$

where $B_t(z)$ is a nonzero polynomial of degree *t*. We must have*M+1+s=3q* (since $p+2r\leq M<M+1$) so that

$$B_t(z)=\frac{27}{4}P(z)R^2(z).$$

Also $t+q=p+q+2r<M+1=3q-s\Rightarrow t+s<2q.$

Differentiating (5) gives

$$3Q^2(z)Q'(z)=27z^M\left((M+1)D_s(z)+zD_s'(z)\right)+B_t'(z)$$
$$\Rightarrow 3zQ^2(z)Q'(z)=27z^{M+1}\left((M+1)D_s(z)+zD_s'(z)\right)+zB_t'(z)$$
$$:=27z^{M+1}\bar{D}_s(z)+\bar{B}_t(z), \qquad (6)$$

where

$$\bar{D}_s(z)=(M+1)D_s(z)+zD_s'(z)\quad \text{(degree } s\text{)}$$
$$\bar{B}_t(z)=zB_t'(z)\quad \text{(degree } t\text{)}$$

From (5) and (6) (eliminating the term in z^{M+1}) we have

$$Q^2(z)\left(\bar{D}_s(z)Q(z)-3D_s(z)zQ'(z)\right)=B_t(z)\bar{D}_s(z)-\bar{B}_t(z)D_s(z). \qquad (7)$$

The left-hand side of (7) either has degree not less than *2q* or is identically zero, while the right-hand side has degree not greater than *t+s<2q*. It follows that

$$\bar{D}_s(z)Q(z)-3D_s(z)zQ'(z)=0=B_t(z)\bar{D}_s(z)-\bar{B}_t(z)D_s(z).$$

Hence

$$\frac{Q'(z)}{Q(z)}=\frac{\bar{D}_s(z)}{3zD_s(z)}=\frac{\bar{B}_t(z)}{3zB_t(z)}=\frac{B_t'(z)}{3B_t(z)}$$

And integrating gives

$$Q(z)=c\sqrt[3]{B_t(z)},\ c\in\mathbf{R}.$$

But

$$\deg\sqrt[3]{B_t(z)}=t/3<q$$

so the result is proved by contradiction.

References

[1] G.A.Baker, Essentials of Padé approximants, New York: Academic Press, 1975.

[2] P. Barone, R. March, ``Some properties of the asymptotic location of poles of Padé approximants to noisy rational functions, relevant for modal analysis," IEEE Trans. Sign. Processing, vol. SP-46, pp. 2448-2457, 1998.

[3] D. Belkic, ``Strikingly stable convergence of the Fast Padé Transform (FPT) for high-resolution parametric and non-parametric signal processing of Lorentzian and non-Lorentzian spectra," Nuclear Instruments and Methods in Physics Research, A Vol. 525, pp. 366-371, 2004.

[4] Y. Bistrits, U. Shaked, ``Discrete multivariable system approximations by minimal Padé-type stable models," IEEE Trans. Circuit Syst., Vol. CAS-31, pp. 382-390, 1984.

[5] R. G. Brooks, A. W. McInnes, ``The existence and local behaviour of the quadratic function approximation," J. Approx. Theory, Vol. 62, pp. 383-395, 1990.

[6] E. Celik, ``On the numerical solution of chemical differential-algebraic equations by Pade series," Appl. Math. Comput. Vol. 153, pp. 13-17, 2004.

[7] W. B. Gragg, ``The Padé table and its relation to certain algorithms of numerical analysis," SIAM Rev., Vol. 14, pp. 1-62, 1972.

[8] P. N. Paraskevopoulos, ``Padé-type order reduction of two-dimensional systems," IEEE Trans. Circuit Syst., Vol. CAS-27, pp. 413-416, 1980.

Lei Shi was born in Liaoning, China, in 1963. He received the Master Degree in mathematics from Northeast Normal University of China in 1989, and then joined the Department of Mathematics, Dalian Jiaotong University, China. His research interests include numerical approximation and its application.

Zhi-Min Xu was born in Liaoning, China, in 1964. She received the Master Degree in mathematics from Northeast Normal University of China in 1989, and joined the Department of Mathematics, Dalian Jiaotong University in 2002, China. Her research interests include numerical approximation and its application.