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Monte Carlo simulations in the case of several risk factors: Cholesky decomposition and copulas

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Abstract

This article presents the Monte Carlo method in the context of simulation of stochastic models in finance. Our research aims to make practical use of the main operative techniques of Monte Carlo simulation applied to finance. In fact, for some years, finance specialists describe several phenomena and work out computational methods using mathematical tools which are becoming increasingly sophisticated. Thus, our work focuses on the problem of the options considered the most prominent example of methods of stochastic calculus in finance in terms of pertinence. We describe in this part how to elaborate Monte Carlo simulations in the presence of several risk factor Y.

Keywords: Monte Carlo, Simulation, Finance, Risk Factor, Stochastic.

1 Introduction

This article aims to present an introduction to probabilistic techniques to understand the most common financial models. Thus, our work deals fundamentally with the problem of options that presents a striking example in the methods of stochastic calculus in finance. In the context of simulation of stochastic models, our study focuses on the Monte Carlo method [1]. In fact, when we want to study a system, we use simulation to measure the effects resulting of the change of complex interactions contained in the system in question. However, the accurate representation of phenomena encounters difficulties whose the cause is not explicit calculations. Thus, the simulation techniques [2] allows to approach numerically these calculations. In this sense, the Monte Carlo methods [3] are designed to the use of repeated experiments in order to evaluate the amount and solve a deterministic system. These methods are used to calculate the integral and solve partial derivative equations, linear systems and optimization problems.

2 Simulation of non-Gaussian vector with correlated components using a copula

Now it is to simulate a vector with non-Gaussian components $(X_1, X_2, ..., X_n)$. We note F_j law of unconditional distribution (or marginal) x_j. Remember that the marginal law (or unconditional) $F_j(x) = Proba(X_j \leq x)$ is X_j in the absence of any hypotheses about the X_k for $k \leq j$ and the joint law $J(x_1, x_2, ..., x_m) = Proba(X_1 \leq x_1 and.....and X_m \leq x_m)$ differs from simple product $F_1(x_1)F_2(x_2)....F_m(x_m)$ of marginal laws, except in the special case of x_j mutually independents. The use of a Gaussian copula permits to transform a Gaussian vector $(U_1, U_2, ..., U_m)$ in a vector $(X_1, X_2, ..., X_n)$ non-Gaussian, thus to obtain indirectly a representation of the joint law of $(X_1, X_2, ..., X_n)$ and make simulations of the latter vector.

2.1 Theory

2.1.1 Cholesky Decomposition:

It allows the toss of a Gaussian vector X by taking into account the variance-covariance matrix Σ presumed known

$$\Sigma \text{ is symmetrical} \Longrightarrow \Sigma = \Lambda\Lambda'$$
$$X = \mu + \Lambda U$$

2.1.2 Decomposition of copula

It allows the toss of a non-Gaussian vector X whose components are correlated, from the Gaussian vector from the Cholesky decomposition.

2.1.3 Proposal

1. Is some distribution law F and a variable U Gaussian centered reduced. The variable $X = F^{-1}(N(U))$ to F for distribution law.
2. Reciprocally, is a random variable X of any distribution law F. The random variable $U = N^{-1}(F(X))$ is normal, centered reduced.

2.2 Example

2.2.1 Statement

Consider two variables T_1 and T_2 distributed according to exponential laws, the two marginal distributions are respectively: $F_1(t) = Proba(T_1 \leq t) = 1 - \exp(-\lambda_1 t) F_2(t) = Proba(T_2 \leq t) = 1 - \exp(-\lambda_2 t)$ T_1 and T_2 are correlated $\lambda_1\lambda_2 \in [0, 1]$. We will associate with T_1 and T_2 two variables U_1 and $U_2 \rightsquigarrow N(0, 1)$ is linked mutually by a coefficient ρ.

2.2.2 Algorithm

- We first simulate M couples $(U_1^i, U_2^i)_{i=1,....,M}$

$$U_2^i = \rho U_1^i + \sqrt{1-\rho} V^i]_{i=1,...,M}$$

with V distributed $N(0, 1)$ and independent of U_1
. Realize M independent tosses of the two Gaussian U_1^i and V^i
. Calculate U_2^i from the previous relationship
. Place in a table the couples values $(U_1^i, U_2^i)_{i=1,....,M}$

- We associate to each couple (U_1, U_2) the couple (t_1, t_2) such as: $N(U_1) = 1 - \exp(-\lambda_1 t_1)$, is $t_1 = \ln(1 - N(U_1))$ and $t_2 = \ln(1 - N(U_2))$
- (t_1, t_2) from the joint law of (T_1, T_2).

2.2.3 Parameter

- $\Lambda_1 = 5\% and \lambda_2 = 4\%$ - $= 0.34$
Table (figure 1) below reproduces 19 tosses (V, U_1) of this simulation that permits to calculate 19 couples (U_1, U_2), then 19 couples (t_1, t_2) representing the remaining time (in years) before the first faults.

$$t_1 = -\frac{1}{\lambda_1} ln(1 - N(u_1))$$
$$t_2 = -\frac{1}{\lambda_2} ln(1 - N(u_2))$$

	v	u1	u2	N(u1)	N(u2)	t1	t2
1	-0.172311	-0.163573	-0.21766	0.4350338	0.4138468	7.2295402	9.5790091
2	-0.775471	-0.86153	-1.022193	0.194473	0.1533448	14.222813	20.35827
3	0.0314542	-0.862237	-0.26358	0.1942785	0.3960517	14.231504	10.056204
4	-0.735327	0.9419939	-0.371243	0.8269021	0.3552284	1.650918	11.237307
5	0.0074889	-0.042628	-0.007451	0.482999	0.4970276	6.321076	7.5904875
6	-0.484692	0.0498023	-0.438884	0.51986	0.3303729	5.6822715	12.02489
7	0.0425243	0.4366689	0.1884583	0.6688242	0.5747413	3.4937598	6.0131895
8	0.5276528	0.6775621	0.7265892	0.7509753	0.7662612	2.4874869	2.8905794
9	-0.057749	0.9910145	0.2826364	0.8391608	0.6112722	1.5230967	5.3441339
10	-0.872267	-0.464593	-0.978264	0.3211116	0.163972	9.8668814	19.63076
11	-0.518418	-0.449214	-0.640266	0.3266386	0.2609998	9.7186494	14.583997
12	1.3150044	0.6042655	1.4421139	0.7271664	0.9253649	2.7673234	0.8421748
13	-0.39427	-0.037844	-0.383648	0.4849062	0.3506196	6.2868459	11.379097
14	-1.089495	-0.731793	-1.273398	0.2321475	0.1014384	12.684719	24.844939
15	-0.349223	0.3198405	-0.219673	0.6254554	0.4130631	4.0760732	9.5995909
16	-1.854568	0.2087802	-1.673098	0.5826901	0.047154	4.6912475	33.162026
17	-0.341846	-0.114164	-0.360297	0.4545538	0.3593125	6.8482948	11.113191
18	0.3300901	-0.031635	0.2996692	0.4873816	0.6177853	6.2426177	5.2290615
19	0.5996075	-1.228764	0.1461063	0.1095801	0.5580812	19.205369	6.3325642

Figure 1: Simulation of non-Gaussian vector

3 Simulation trajectories

3.1 Theory

The simulated trajectories of random vector of m factors $\overrightarrow{Y}(t)$ between 0 and T must take into account the structure as expressed by the variance-covariance matrix $\sigma(t)$ of variables in the case of m factors. Depending on whether a representation of the dynamics of $\overrightarrow{Y}(t)$ in continuous time (equation: $dY = \mu(t, Y(t))dt + \sigma(t, Y(t))dw$ Where dw is the increment of a standard Brownian motion and $\mu()$ and $\sigma()$ are two known functions) or discrete time (equation: $Y(t_j) - Y(t_{j-1}) = \mu(t_{j-1}, Y(t_{j-1}))\Delta t + \sigma(t_{j-1}, Y(t_{j-1}, Y(t_{j-1}))\sqrt{\Delta t} U_j$ Where u_j a particular embodiment of $U(j)$ (for $j = 1, ...N$). The $U(j)$ are normal variables, centered, reduced and independently distributed) is used, the general term of the matrix $\Sigma(t)$ is:

$\sigma'_{ij}(t, \overrightarrow{Y}(t)) = \frac{1}{dt} cov(dY_i; dY_j)$ pour $i, j = 1, ., m$ in continuous time ;
$\sigma_{ij}(t, \overrightarrow{Y}(t)) = \sigma'_{ij}(t, \overrightarrow{Y}(t))\Delta t$ et
$\sigma_i = \sqrt{\sigma_{ii}} = \sigma'_i \sqrt{\Delta t}$ in discrete time .

Exceptionally, premiums here designate annualized parameters to distinguish from parameters corresponding to periods of time Δt. As Monte Carlo simulations require discretization of continuous processes, this is the second formulation (in discrete time) which is used in practice. In addition, we note $\rho_{ij}(t, \overrightarrow{Y}(t)) = \frac{\sigma'_{ij}}{\sigma_i \sigma_j}$ the correlation coefficient between variables of factors i and j: $\overrightarrow{\mu}(t, \overrightarrow{Y}(t))$ designates the drift of $\overrightarrow{Y}(t)$ between t and $t + \Delta t$. We decompose the period $(0, T)$ into N periods of time $\Delta t = \frac{T}{N}$ noted $(t_{j-1}, t_j)(j = 0, ..., N, t_0 = 0 and t_N = T)$ and we express the variations of the first component Y_1 between t_{j-1} and t_j by using the discretized equation of type (2)

$$Y_1(t_j) - Y_1(t_{j-1}) = \mu(tj - 1, \overrightarrow{Y}(t_{j-1})) + \lambda_{11}(t_{j-1}, \overrightarrow{Y}(t_{j-1}))U_1(j), \text{ for } j = 1, ., N$$

where the variables $U_1(j)$ are normal, centered, reduced and serially independents. The variations of different Y must be mutually correlated. To represent them, we use a triangular Cholesky decomposition as was shown previously that, in this context, leds to ask:

$$\begin{aligned}
&Y_1(t_j) - Y_1(t_{j-1}) = \mu_1(.) + \lambda_{11}(.)U_1(j)\\
&Y_2(t_j) - Y_2(t_{j-1}) = \mu_2(.) + \lambda_{21}(.)U_2(j) + \lambda_{22}(.)U_2(j)\\
&\vdots\\
&Y_i(t_j) - Y_i(t_{j-1}) = \mu_i(.) + \lambda_{i1}(.)U_i(j) + \ldots + \lambda_{ii}(.)U_i(j)\\
&\vdots\\
&Y_m(t_j) - Y_m(t_{j-1}) = \mu_m(.) + \lambda_{m1}(.)U_m(j) + \ldots + \lambda_{mm}(.)U_m(j)
\end{aligned}$$

Where the $U_i(j)$ are normal variables, centered, reduced and serially independents and mutually independents and where, to simplify the writing, (.) is $(t_{j-1}, \overrightarrow{Y}(t_{j-1}))$. Two trajectories of N points for $\overrightarrow{Y}$ are obtained from MXN tosses U and their antitheticals. According to the procedure explained, the $\lambda_{ij}(.)$ are determined step by step (starting with λ_{11}) from the variance-covariance matrix $\Sigma(.)$ of general term σ_{ij}, and are solutions of i the following equations: $\sum_{j=1}^{i} \lambda_{ij}^2 = \sigma_{ij}$, to comply with the presumed variance of $Y_i(t_j) - Y_i(t_{j-1})$, for $k = 1, ..., i-1$, to respect the $i - 1$ covariances between the variation of $Y_i(t)$ and the one of $Y_K(t)$.

3.2 Example: Simulations of a three-factor model (prices, rates and stochastic volatility)

Now consider the case of a position depending on three factors:
- Price action S
- Interest rate r
- Volatility *sigma* of the action that influences the dynamics of S and of the price action S_T

3.2.1 Statement

Consider a portfolio composed of TCN, an action (or an index) and options on the action whose value depends on the three previous random factors, these three factors are presumed to follow the following three-dimensional diffusion process
- $dr = a(b - r(t))dt + \lambda_{11}dw_1$ (r)
- $\frac{dS}{S} = (r(t) + \theta)dt + \lambda_{21}(.)dw_1 + \lambda_{22}(.)dw_2$ (S)
- $d\sigma = c((t) - \sigma(t))dt + \lambda_{31}(.)dw_1 + \lambda_{32}(.)dw_2 + \lambda_{33}(.)dw_3$ (σ)

where w_1 and w_2 w_3 are independent standard Brownian. These three equations (r) (S) and (σ) require explanation. Equation (r): the rate $r(t)$ follows a Orstein-Uhlenberck process with a force of retraction force a that brings it to a normal value b. λ_{11} = annualized standard deviation σ_r of rate variations. - A, b and σ_r are assumed to be constant and known. Equation (S): the return equation $\frac{dS}{S}$ of the action. - Θ = The assumed constant risk premium. - $(R(t) + \theta)$: The expected return of the action - (.) (.) Λ_{21} and λ_{22} must be compatible with the volatility $\sigma(t)$ of the action following the process represented by the third equation. - $\Lambda_{21}^2 + \lambda_{21}^2 = \sigma^2(t)$ (C1) *and* $\lambda_{(.)(.)21} = \rho_1$ *sigma*(t) (C2) *Theconditions* (C1) *and* (C2) *leadtherefore to values of* λ_{21} and λ_{22} $\lambda_{21} = \rho_1\sigma(t)$ and $\lambda_{22} = \sigma(t)\sqrt{1-\rho^2}$ Equation of (σ): the volatility $\sigma(t)$ follows a process involving a retraction force towards a normal value $\phi S(t)$ which depends negatively on the level of price S. The coefficients λ_{31} λ_{32} and λ_{33} allow a correlation with r and S
- $\lambda_{31} = 0$
- $\lambda_{32}^2 + \lambda_{33}^2 = k^2$
- $\lambda_{32}\lambda_{33} = \rho_2 k\sigma$

So we can deduce $\lambda_{32} = \frac{\rho_k 2}{\sqrt{1-\rho_1^2}}$ and $\lambda_{33} = k\sqrt{\frac{1-\rho_1^2-\rho_2^2}{1-\rho_1^2}}$
The diffusion process governing the three factors that respects volatilities and correlations desired is then:
- $dr = a(b - r(t))dt + \sigma_r$ dw$_1$
$-$dS S $=$ (r (t) $+ \theta)dt + \rho_1\sigma(t)dw_1 + \sigma(t)\sqrt{1-\rho^2}dw_2$
- $d\sigma = c(\phi S(t) - \sigma(t))dt + \frac{\rho_k 2\sqrt{1-\rho_1^2}}{d}w_2 + k\sqrt{\frac{1-\rho_1^2-\rho_2^2}{1-\rho_1^2}}dw_3$

This three-dimensional process is written in continuous time where the Monte Carlo simulation is based on a discretization of this process: $r(t_j) - r(t_{j-1}) = a(b - r(t_{j-1}))\Delta t + \sigma_r$ U$_1$
S $(t_j) - S(t_{j-1}) = S(t_j)[(r(t_{j-1}) + \theta)\Delta t + \rho_1\sigma(t_{j-1})U_1 + \sigma(t_{j-1})\sqrt{1-\rho^2}U_2]$
$\Sigma(t_j) - \sigma(t_{j-1}) = c(\phi S(t_{j-1}) - \sigma(t_{j-1}))\Delta t + \frac{\rho_k 2\sqrt{1-\rho_1^2}}{U}{}_2 + k\sqrt{\frac{1-\rho_1^2-\rho_2^2}{1-\rho_1^2}}U_3$

3.2.2 Parameter values

For a numerical application we can choose, for example, the following parameters:
- The steps are weekly $\Delta t = 1/52 = 0.1923$
- $a = 0.18, b = 0.04, \sigma_r = 0.2, \theta = 0.05, \rho_1 = 0.3, c = 0.5, \rho_2 = -0.2; k = 0.08$ We can choose $\phi(S(t))$ equal to a constant (e.g. 0.3) or a decreasing function of $S(t)$, for example: $\phi(S(t)) = 0.15 \times (1 + (\frac{2S(0)}{S(t) - S(O)})^2)$, the "normal" annualized volatility is therefore equal to 0.3 for a price $S(t) = S(0)$ actual.

3.2.3 Generate the variables $U_i]_{i=1,2,3}$

Using the Box-Muller method we generate the variables $U_i]_{i=1,2,3}$
$U_i = \sqrt{-2 * LOG(RAND())} * \cos(2 * PI()RAND())$

	1	2	3	4	5	6	7	8
U1	-0.08870399	-1.09626975	-0.0023183	-1.59701802	0.16559002	-0.91015102	-0.12350611	0.26724561
U2	0.30303392	0.93890331	-0.56684643	-0.28471401	0.29177299	0.15621329	0.73121925	-0.00232058
U3	-0.75844839	1.27600271	-0.24066881	0.43502512	0.12401024	0.45747992	0.34696201	-0.82118252

Figure 2: Generate the variables $U_i]_{i=1,2,3}$

3.2.4 Calculation of three factors $r(t_j), S(t_j)$ *and* $\sigma(t_j)$

Two tosses of 52 weekly points are obtained from 152 tosses of U and their antitheticals; the simulation of 1000 trajectories requires therefore 78,000 tosses in one year (counting U_j and its antithetical to one toss).

r(ti)						
24	25	26	27	28	29	30
0,00007843	0,01069429	-0,01437294	-0,00585446	-0,00316702	-0,01697580	0,00271403
0,07992157	0,06930571	0,09437294	0,08585446	0,08316702	0,09697580	0,07728597

Figure 3: Calculation of interest rate
$r(t_j) - r(t_{j-1}) = a(b - r(t_{j-1}))\Delta t + \sigma_r U_1$

S(ti)						
24	25	26	27	28	29	30
104,907346	104,444620	105,530580	105,057691	105,103481	105,809770	105,017939
103,153857	103,998949	103,307029	104,090296	104,402825	104,088761	105,146076

Figure 4: Calculation of the price action $S(t_j) - S(t_{j-1}) = S(t_j)[(r(t_{j-1}) + \theta)\Delta t + \rho_1\sigma(t_{j-1})U_1 + \sigma(t_{j-1})\sqrt{1-\rho^2}]U_2$

sigma(ti)						
24	25	26	27	28	29	30
0,06456451	0,06980259	0,07352351	0,07527249	0,07699572	0,08144971	0,08906773
0,06841146	0,06766399	0,06839051	0,07104620	0,07368530	0,07355168	0,07021249

Figure 5: Calculation of the volatility of the action $\sigma(t_j) - \sigma(t_{j-1}) = c(\phi S(t_{j-1}) - \sigma(t_{j-1}))\Delta t + \frac{\rho_2 k}{\sqrt{1-\rho_1^2}}U_2 + k\sqrt{\frac{1-\rho_1^2-\rho_2^2}{1-\rho_1^2}}U_3$

3.2.5 Graph of the action price depending on time

Figure 6 illustrates the graph of the action price depending on time (step = Weekly)

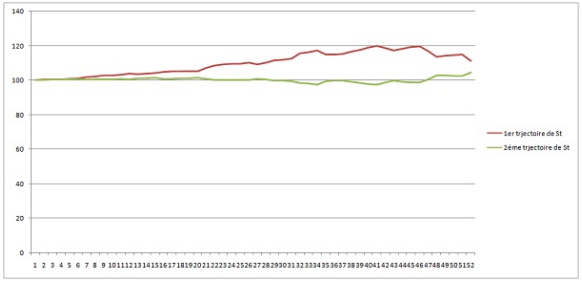

Figure 6: Graph of the action price depending on time

4 Conclusion

Our work focuses on the study of a generalization of a position whose the value $V(t, \overrightarrow{Y}(t))$ depends on m risk factors $\overrightarrow{Y} = (Y_1, Y_2, ..., Y_m)$, that are often correlated; so we can not simulate the realizations of the different factors $Y_i(t)$ independently of each other. Indeed, we have distinguished the case where the simulation trajectories of $\overrightarrow{Y}(t)$ between 0 and T where only terminal values $\overleftarrow{Y}(t)$, $V(t, \overrightarrow{Y}(t))$ are necessary. thus, we must distinguish the case of a Gaussian vector $\overrightarrow{Y}(t)$ of non-Gaussian case. The toss of a Gaussian vector $\overrightarrow{Y}(t)$ whose components are correlated may be based on the Cholesky decomposition. Copulas are used to represent and simulate the realizations of non-Gaussian vector with correlated components.

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