Observations on the effectiveness of procedures and some improvement techniques

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Abstract

We will describe in this article a method allowing the simulation of financial models. This method is often useful in the context of financial mathematics, because it allows calculating the price of any option as long as we know to express it in the form of the expectation of a random variable that we simulate. In this case, the Monte Carlo method described later allows then writing quickly an algorithm to evaluate this option and it is often very greedy in time calculation. Effective procedure leads to a sufficient precision at the cost of a limited calculation time. To identify more precisely the efficiency notation and indicate some techniques to increase it, we assume that we want to estimate by Monte Carlo simulations, a parameter e of the distribution of $V(T, \overrightarrow{Y}(t))$. It can be a quantile of this distribution (that is the case when it comes to appreciating a VaR), of an expectation (for example if we try to evaluate an option) or of any time of the distribution $V(T, \overrightarrow{Y}(t))$. On the other hand, Monte Carlo simulations, in the standard form is not suitable for the evaluation of american options. The reason is that the opportunity to exercise at any time requires to calculate at each date and each trajectory a conditional expectation, thus to remove trajectories at each step. This is infeasible in practice, even if it is limited to a finite number of possible exercise dates. Thus, we use the techniques of improving the adaptation of Monte Carlo method to make it suitable for the evaluation of american options.

Keywords: Monte Carlo, Financial models, Precision, Computation time.

1 Introduction

In most applications, a compromise must be made between two antinomical objectives: precision and richness of empirical information obtained (that is desired maximum), which increases with the number of simulations carried out; the calculation time (that is desired minimum). Consider the case of a simple Monte Carlo[1] that integrates sophisticated techniques. Call M the number of simulations that lead to an estimator $\hat{e}$ of the parameter e. The standard error of $\hat{e}$, equal to the square root of the variance of $\hat{e}$ divided by M, is inversely proportional to $\sqrt{M}$: the number of simulations should therefore be quadrupled if we want to double the precision of $\hat{e}$ (when this one is measured with the standard error). However, several techniques[2] can be implemented to improve the effectiveness of simulations. The Monte Carlo simulations, in the standard form are not suitable for the evaluation of american options[3]. The reason is that the trajectories are separated and the induction reverse (backward) of the value (i.e. the calculation of an updated expectation from T to 0), possible in the case of a tree, is not realizable in the case of a beam trajectories dissociated. From then on, at a time $t_i < T$, the simulation cannot allow to simply determine the value of continuation which must be compared to the intrinsic value in order to decide to exercise the option or keep it. This is why the two competing digital methods, trees and finite difference methods, have been considered for a long time the only ones capable of american option valuation. However, since the mid-90s, adaptations of the Monte Carlo method also exist to make it suitable for the evaluation of these options. The most used method was originally proposed by Carrire [4](1996), then improved, developed and popularized by Longstaff and Schwartz[5] (2001). It couples least squares regression to a Monte Carlo simulation.

2 Precision, computation time and some techniques for variance reduction

Monte Carlo simulations are often very gourmand in computation time. In fact, in most applications, a compromise must be made between two antinomic objectives: the precision and the richness of the empirical informations obtained (that is to maximum), which increase with the number of operated simulations, the calculation time (that is to minimum). An effective procedure leads to sufficient precision at the cost of a limited computation time. To delimite more precisely this notation of the effectiveness and indicate some techniques to increase it, we assume that we want to estimate by Monte Carlo simulations, a parameter e of the distribution of $V(T, \overrightarrow{Y}(t))$. It can be a quantile of this distribution, an expectation or any moment the distribution of $V(T, \overrightarrow{Y}(t))$. However, several techniques to reduce the variance can be implemented to improve the effectiveness of simulations. We quote two techniques: antithetical variables and control variables.

2.1 Antithetical variables

It is based on the principle of associating to each toss U_i its opposite $-U_i$, but we must check that their average equal to 0, and variance equal to 1. If this is not the case: we subtract the average and we divide by the standard deviation to standardize the sample.

2.2 Control variable

The e parameter to estimate is the expectation of $V(T, S(T))$. Suppose that there is a $V'(T, S(T))$ highly correlated with $V(T, S(T))$ whose expectation $e' \approx e$ (although it is different) and e is known. Instead of estimating e by using the standard estimator $\hat{e} = \frac{1}{M}\sum_{i=1}^{M} V(T, \overrightarrow{Y})$, we will estimate the difference $\delta = e - e'$ by using simulation on

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$V - V'$. Therefore, in its simplest form, the method of the control variable is as follows:

. We calculate from N simulations Y_i:$\hat{\delta} = \frac{1}{M}\sum_{i=1}^{M}(V(T,\overrightarrow{Y_i}) - V'(T,\overrightarrow{Y_i}))$

. The estimator used for e will be $\hat{e}_1 = e' + \hat{\delta}$ (instead of $\hat{e}$ standard).

3 Examples

3.1 Example 1 : Evaluation of Path-Dependent option

3.1.1 Statement

Suppose a Monte Carlo simulation is to evaluate a purchase path-dependent option. The estimateion of the latter requires simulations "tightened" throughout the trajectory of the underlying, very time-consuming calculation. Suppose the European call option is evaluated by Black-Scholes[6] . The price e' of the European option is therefore known and equal to the risk-neutral expectation of payoffs $V'(T,S(T))$ updated: $e' = exp(-rT)E(V'(t,S(t)))$. The price e of path-dependent exotic option is unknown, slightly more inferior than e' and equal to the present value of the risk-neutral expectation of the payoff induced. These payoff depend on the trajectory of $S(t)$. Remember that the simple method of Monte Carlo consists in simulating M trajectories of N points to $S(t)$, in associating with each i simulated trajectory $(S_1^i, S_2^i, ..., S_N^i)$ an updated payoff $\psi(S_1^i, S_2^i, ..., S_N^i)$ and in estimating the price of this option by the average $\hat{e} = \frac{1}{M}\sum_{i=1}^{M}\psi(S_1^i, S_2^i, ..., S_N^i)$ The use of the control variable $(e', V'(t,S(t)))$ consists in estimating the difference $\delta = e - e'$ by using the empirical average. $\hat{\delta} = \frac{1}{M}\sum_{i=1} M\psi(S_1^i, S_2^i, ..., S_N^i) - \exp(-rT)E(V'(T,S(T)))$ and in using $\hat{e}_1 = e' + \hat{\delta}$ as an estimator of the price of the exotic option.

3.1.2 Data of example

	A	B	C	D
1	Exemple 6			
2				
3	S(0)=	100		
4	volatilité(sigma)=	36%	0,049923018	
5	taux d'interet(r) =	12%	0,002307692	
6	maturité (T)=	0,1923		
7	hebdomadaire	52		
8	pas=	0,0036981		
9				
10	S(n)=S(0)exp((mu-0,5sigma²)t+sigma*wt)			
11				
12	Strike (K)=	102		
13	Barrière d'activation=	95		

Figure 1: Data of Example 1

3.1.3 Generate the $S(t_i)$

E	F	G	H	I	J	K	L	M	N	O
	S(ti)									
	1	2	3	4	5	6	7	8	9	10
1	101,971386	100,95753	99,3331305	100,725453	98,7752388	99,3734831	99,8968768	97,2095073	93,3641692	91,4262923
2	98,2751509	99,4730339	101,314592	100,126476	102,320373	101,920544	101,602027	104,632736	109,173725	111,724726
3	101,069105	100,678898	98,3900425	102,162689	97,9357107	96,5173173	99,207587	103,152841	102,281061	106,820448
4	99,1524893	99,7483284	102,285712	98,717886	103,197487	104,936603	102,307954	98,6041361	99,6559289	95,6238032
5	101,925384	109,478286	101,08528	101,721281	96,5690088	98,6023538	98,5535183	102,9378	103,390096	103,544136
6	98,3195056	91,7309927	99,558468	99,1462615	104,658	102,717623	102,986939	98,8101234	98,586949	98,6495025
7	104,067904	102,617247	101,109917	108,045426	111,711455	114,030348	111,712347	118,416555	119,527568	123,283193
8	96,2953314	97,8641715	99,5342084	93,3430048	90,4716469	88,8202094	90,8558941	85,8942127	85,2766793	82,8545824
9	104,975693	104,076432	97,2364155	103,878773	103,853739	103,357973	101,911291	101,695603	102,730811	100,227529
10	95,4626068	96,492084	103,499245	97,0870605	97,3168548	97,9914673	99,5937261	100,017075	99,2196407	101,913891
11	101,731591	99,4861008	97,9284717	100,99781	104,75945	106,63847	107,178988	108,227049	106,718393	107,086774
12	98,5067981	100,94427	102,76782	99,8564696	96,4754897	94,9769759	94,6988337	93,9810962	95,5122528	95,3859851
13	99,1418652	96,8885105	98,8776649	99,4879976	99,8235518	102,568393	102,44636	100,319291	102,208375	94,5873141
14	101,079935	103,650596	101,781283	101,371873	101,245839	98,7458135	99,0735567	101,389241	99,7267994	107,990988
15	107,451997	105,459803	111,020413	115,158071	112,243438	113,897241	112,070704	112,66654	111,058288	107,776425
16	93,2626062	95,2263468	90,6490557	87,5777497	90,0428521	88,9240099	90,5653742	90,2778829	91,7798604	94,7756193
17	104,215485	102,653921	106,606603	106,67743	109,975938	117,387111	116,967278	108,411076	106,245942	105,100757
18	96,1589662	97,8292081	94,4021789	94,5400043	91,8993688	86,2803365	86,7740564	93,8215641	95,9369734	97,1884286
19	96,9888709	98,0794199	98,4338449	100,047256	101,283748	103,48651	103,349067	104,843923	107,196374	107,420982
20	103,323745	102,392039	102,240196	100,80521	99,78619	97,8697554	98,2081934	97,0136988	95,0863705	95,089221

Figure 2: Generate the $S(t_i)$

3.1.4 Calculation of payoff"Call" et ψ

Q	R	S	T	U	V	W	X	Y	Z	AA	AB
	V(ti,S(ti)) "Call"										
	1	2	3	4	5	6	7	8	9	10	ψ(S1,S2,.....S10)i
1	0	0	0,12969197	5,02749134	6,56131262	8,27784242	12,0613815	13,4059231	13,6399267	14,6567158	0
2	0,38312178	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0,11753008	3,51032614	7,82679857	4,8193227	3,61497308	4,99023242	0
5	4,44474453	0	1,69324981	0	0	0	0	0	0	3,60809042	0
6	0	0	0	0	1,89289318	2,69199579	3,10682724	4,2574005	0,48642628	0	0
7	0	0	0	0	0	0	0	4,20845608	7,41081399	8,02375485	8,02375485
8	0	3,00422105	0	4,89257275	3,02455314	3,55211839	0,89992525	0	0	0	0
9	0	0	0	1,82222527	0	4,1026611	3,83138298	5,17216758	8,46331513	8,37282736	0
10	1,98673118	0	0	0	0	0	0	0	0	0	0
11	4,19889315	6,85839594	6,20829632	7,25104905	0	0	0	0	0	0	0
12	0	0	0	0	0	1,72444349	2,16900793	4,94553621	9,58967862	14,7408705	14,74087046
13	0,40381674	6,29120958	9,07340754	12,2342785	15,7377822	16,6416835	18,3511239	21,7825573	24,0563868	29,6835553	0
14	0	0	0	0	0	0	0	0	0	0	0
15	0,32608293	2,83137847	12,3795361	17,2481134	15,9896158	17,0962854	20,1875735	22,5512139	21,4274637	16,871929	0
16	0	0	0	0	0	0	0	0	0	0	0
17	1,42682335	2,88332364	4,70802346	10,5203596	0	1,93844933	0	0	0	0	0
18	0	0	0	0	0	0	3,00626231	5,27206875	1,50580019	3,27755275	3,277552749
19	0	0	0	0	0	0	0	0	0	0	0
20	0,6487048	1,97427635	4,33385057	0	3,6968132	5,52443304	6,16869351	4,41014287	2,45158408	0	0
21	0	0	4,31510945	4,78092768	1,68245952	4,23671812	6,3615894	8,51820592	10,940647	15,2288059	0
22	0	0	0	0	0	0	0	0	0	0	0

Figure 3: payoff "Call"$(V(t;S(t)))$ and ψ

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3.1.5 Calculation of payoff"Put" et ψ

AD	AE	AF	AG	AH	AI	AJ	AK	AL	AM	AN	AO
	V(ti,S(ti)) "Put"										
	1	2	3	4	5	6	7	8	9	10	ψ(S1,S2,....S10)
1	4,12006483	2,6459701	0	0	0	0	0	0	0	0	0
2	0	0,92154525	3,45965017	7,7692134	8,903122	10,1574395	13,0152353	13,8650372	13,856446	14,4390106	14,4390106
3	2,19643206	1,92073463	2,62849105	1,11712417	3,0285573	6,00750539	9,5842569	6,78036667	5,48987621	6,52794406	6,527944063
4	1,59022979	1,65402146	0,72453703	2,02976607	0	0	0	0	0	0	0
5	0	0,115455	0	0,45296432	4,7198207	5,25717513	5,43419672	6,27681561	2,54376472	0	0
6	7,85487929	3,43203942	4,94550906	2,68361298	0	0	0	0	0	5,27845704	5,278457042
7	3,65083511	6,36049413	3,01413893	7,65027637	5,76803352	6,04551261	3,3631431	0	0	0	0
8	0,10535418	0	0,32996889	0	0	0	0	6,23270316	8,8381474	9,160254	9,160253997
9	5,62949957	2,9403296	2,1680462	0	0,42782964	0	0	0	0	0	0
10	0	0,62118686	1,19163964	4,86006036	2,49716137	6,54339874	6,09531732	7,09386	9,72579916	9,45387521	9,453875208
11	0	0	0	0	1,24772451	4,35474809	4,56482802	6,89274166	10,6571962	14,5021305	14,50213045
12	7,6369325	9,74664895	8,99515733	9,68706769	1,68743408	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0
14	4,13984555	9,26346253	11,3941944	13,7140246	16,1590823	16,6320323	17,6657215	19,8293214	21,1400421	24,4308803	24,43088035
15	0	0	0	0	0	0	0	0	0	0	0
16	4,06550437	6,20280681	14,0131457	17,426044	16,3422986	16,9578899	18,9332453	20,3364303	19,4177787	16,0707353	16,07073529
17	0	0	0	0	0,51930564	0	5,34171526	7,18224493	3,5232566	4,97478249	4,974782488
18	5,1077916	6,25025193	7,68754581	12,3692458	2,40746823	4,55579644	0	0	0	0	0
19	4,37330961	5,41311171	7,35567441	2,30634304	6,38009462	7,80560039	8,16762695	6,4142178	4,41493975	1,75668848	1,756688476
20	0	0	0	0,83724799	0	0	0	0	0	0,10215465	0
21	2,4283559	0,89348015	0	0	0	0	0	0	0	0	0
22	1,35635378	2,67354875	7,33899061	7,55162879	4,52238109	6,66385264	8,33465919	9,96722897	11,7498164	14,8663184	14,86631837

Figure 4: payoff "Put"$(V(t;S(t)))$ and ψ

3.1.6 the estimated premium for a European option and path dependent option

	A	B	C	D
15	Pour un Call			
16				
17	La prime estimée pour une option européenne			
18				
19	ê =	0,343544		
20				
21	La prime estimée pour une option path-dependent			
22				
23	ê' =	3,6810682		
24				
25	alors l'écart δ =		3,33752422	
26				
27	Pour un Put			
28				
29	La prime estimée pour une option européenne			
30				
31	ê =	3,8396689		
32				
33	La prime estimée pour une option path-dependent			
34				
35	ê' =	4,1228448		
36				
37	alors l'écart δ =		0,28317586	

Figure 5: the estimated premium for a European option and path-dependent

3.2 Example 2: Monte Carlo and American options

3.2.1 Description

The idea consists in estimating at each date t_j to which S takes the value S_j a continuation value $V_c(t_j;S_j)$. This value may be compared to the exercise value $V_e(S_j)$ (or intrinsic value) to decide whether to exercise or to continue. The value of an American option dated t_j is denoted $O(t_j;S_j)$. It is such that:

$$\mathrm{O}\ (t_j;S_j) = Max[V_c(t_j;S_j)V_e(S_j)]$$

$\mathrm{V}_c(t_j;S_j) = \exp(-r\Delta t)E[O(t_{j+1})S(t_{j+1})|i_j]$
$\Delta t = t_{j+1} - t_j = T/N$ is the time step of the simulation, $E[.|I_j]$ is the risk-neutral conditional expectation, developed with the information I_j in time t_j. In the case of an option whose price depends only on the final value of a single underlying, we write:

$\mathrm{V}_c(t_j;S_j) = \exp(-r\Delta t)E[O(t_{j+1})S(t_{j+1})|S(t_j) = S_j]$ As this mode is selected, the option price can be determined by getting back in time t_n 0, in the spirit of the method of standard trees.

3.2.2 Estimation of the continuation value by regression (Carrire, Longstaff and Schwartz)

The determination of the conditional expectation, so the continuation value can be based on a regression made with $q+$ 1 regressors: 1, $\Phi_1(S_{n-1})$........$\Phi_q(S_{N-1})$:
$E[O(t_{j+1}, S(t_{j+1})|S(t_j) = S_j] = a_j + \sum_{k=1}^{q} \beta_{k,j}\Phi_k(S_j)$
The techniques differ in the number q of explanatory variables, the form of the functions ϕ_i chosen and the estimation method.

	A	B	C	D	E	F	G	H	I	J
1	Exemple 7									
2						0	1	2	3	valeur terminal du Put
3					1	1	1,15	1,23	1,18	0,00
4	volatilité(sigma)=	36%			2	1	0,97	1,01	1,18	0,00
5	rentabilité(mu)=	12%			3	1	1,06	1,05	1,55	0,00
6	maturité (T)=	3			4	1	1,05	1,19	0,90	0,20
7					5	1	0,80	0,88	0,84	0,26
8					6	1	1,40	1,42	1,65	0,00
9					7	1	1,40	1,36	1,37	0,00
10	S(n)=S(0)exp((mu-0,5sigma²)t+sigma*wt)				8	1	0,80	0,92	1,02	0,08
11					9	1	1,04	1,23	0,95	0,15
12	Strike (K)=	1,1			10	1	1,07	1,02	1,47	0,00

Figure 6: The trajectories of the price of the underlying and the terminal value of the put

	Put Américain				
	0	1	2	3	valeur terminal du Put
1	1	1,09	1,08	1,34	0,00
2	1	1,16	1,26	1,54	0,00
3	1	1,22	1,07	1,03	0,07
4	1	0,93	0,97	0,92	0,18
5	1	1,11	1,56	1,52	0,00
6	1	0,76	0,77	0,90	0,20
7	1	0,92	0,84	1,01	0,09
8	1	0,88	1,22	1,34	0,00
9	1	1,15	1,12	1,00	0,10
10	1	1,10	1,12	1,15	0,00

Figure 7: Americans put

Multiple linear regression is: $Y(Payoff) = \alpha_0 + \alpha_1 X_1(S(t=2)) + \alpha_2 X_2(S(t=1))$

S(2)	S²(2)	P3
1,08	1,1664	0
1,07	1,1449	0,07
0,97	0,9409	0,18
0,77	0,5929	0,2
0,84	0,7056	0,09

Regression

trajectoire	Prévisions pour P3
1	0,03901247
3	0,048736538
4	0,124794274
6	0,161366459
7	0,166090259

Figure 8: Prevision of payoffs

trajectoire	S2	Ve(2)	Vc(2)	max(Vc(2);Ve(2))	
1	1,08	0,02	0,0367	0,0367	On continue
3	1,07	0,03	0,0459	0,0459	On continue
4	0,97	0,13	0,1175	0,13	exercice anticipé
6	0,77	0,33	0,1520	0,33	exercice anticipé
7	0,84	0,26	0,1564	0,26	exercice anticipé

Figure 9: Analysis of the exercise decision at t = 2

Table 10 below represents the payoffs of the option in the hypothesis of an exercise possible only at the dates 2 and 3 and their corresponding present value at date 2 of the 10 trajectories.

trajectoire	valeur presente t=2	Payoff 2	Payoff 3
1	0,0000		
2	0,0000		
3	0,0659		0,07
4	0,7700	0,13	
5	0,8400		
6	1,0800	0,33	
7	1,0700	0,26	
8	0,9700		
9	0,7700		0,10
10	0,8400		

Figure 10: Payoffs of the option in the hypothesis of a possible exercise

Multiple Linear Regression

S(1)	S²(1)	P2
1,09	1,1881	0
0,93	0,8649	0,13
0,76	0,5776	0,33
0,92	0,8464	0,26
0,88	0,7744	0

Regression

trajectoire	Prévisions pour P2
1	0,014318978
4	0,11547396
6	0,303753933
7	0,124244717
8	0,162208412

Figure 11: Prevision of payoffs

trajectoire	S1	Ve(1)	Vc(1)	max(Vc(1);Ve(1))	
1	1,09	0,01	0,0135	0,0135	On continue
4	0,93	0,17	0,1087	0,17	exercice anticipé
6	0,76	0,34	0,2861	0,34	exercice anticipé
7	0,92	0,18	0,1170	0,18	exercice anticipé
8	0,88	0,22	0,1528	0,22	exercice anticipé

Figure 12: Analysis of the exercise decision at t = 2

Table 13 below represents the payoffs of the option in the hypothesis of an exercise possible only at the dates 2 and 3 and their corresponding present value at date 1 of the 10 trajectories.

trajectoire	payoff 1	Payoff 2	Payoff 3
1			
2			
3			0,07
4	0,17		
5			
6	0,34		
7	0,18		
8	0,22		
9			0,10
10			

Figure 13: Payoffs of the option in the hypothesis of a possible exercise

4 Conclusion

The Monte Carlo method is unfortunately not very effective and we only use it if we do not know to explicit option price in analytical form. Similarly, when we ask ourselves complex questions about a strategy of portfolio management (e.g., what is the law in a month of a hedged portfolio every 10 days in delta neutral), the correct answer is analytically inaccessible. Simulation methods are then necessary.

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