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The $L(2,1)$-Labelings on the Homomorphic Product of two Graphs

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Abstract

The concept of $L(2,1)$-labeling in graph came into existence with the solution of frequency assignment problem. In fact, in this problem a frequency in the form of nonnegative integers is to assign to each radio or TV transmitters located at various places such that communication does not interfere. This frequency assignment problem can be modeled with vertex labeling of graphs. An $L(2,1)$-labeling (or distance two labeling) of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u)-f(v)|\geq 2$ if $d(u,v)=1$ and $|f(u)-f(v)|\geq 1$ if $d(u,v)=2$, where $d(u,v)$ denotes the distance between u and v in G. The $L(2,1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-labeling with $\max\{f(v):v\in V(G)\}=k$. This paper considers the $L(2,1)$-labeling number for the homomorphic product of two graphs and it is proved that Griggs and Yeh's conjecture is true for the homomorphic product of two graphs with minor exceptions.

Keywords: *Channel assignment, $L(2,1)$-labeling, $L(2,1)$-labeling number, Homomorphic product of two graphs.*

1. Introduction

The frequency assignment problem asks for assigning frequencies to transmitters in a broadcasting network with the aim of avoiding undesired interference**.** Hale [21] was first person who formulated this problem as a graph vertex coloring problem. According to Roberts [8], In order to avoid interference, any two "close" transmitters must receive different channels and any two "very close" transmitters must receive channels that are at least two channels apart. To translate the problem into the language of graph theory, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent and "close" if they are of distance two in the graph. Based on this problem, Griggs and Yeh [11] considered an $L(2,1)$-labeling on a simple graph. An $L(2,1)$-labeling (or distance two labeling) of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u)-f(v)|\geq 2$ if $d(u,v)=1$ and $|f(u)-f(v)|\geq 1$ if $d(u,v)=2$, where $d(u,v)$ denotes the distance between u and v in G. A $k-L(2,1)$-labeling is an $L(2,1)$-labeling such that no label is greater than k. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$ or λ, is the smallest number k such that G has a $k-L(2,1)$-labeling. The $L(2,1)$-labeling has been extensively studied in recent past by many researchers [see 2, 5, 9, 10, 12, 13, 20]. The common trend in most of the research paper is either to determine the value of $L(2,1)$-labeling number or to suggest bounds for particular classes of graphs.

Griggs and Yeh [11] provided an upper bound of $\lambda(G)$ is $\triangle^2+2\triangle$ for a general graph with the maximum degree $\triangle$. Later, Chang and Kuo [9], improved the bound to $\triangle^2+\triangle$, while Kral and Skrekovski [3] reduced the bound to $\triangle^2+\triangle-1$. Furthermore, recently Gonccalves [2] proved the bound $\triangle^2+\triangle-2$ which is the present best record. If G is a graph of diameter 2 then $\lambda(G)\leq\triangle^2$. The upper bound is attainable for Moore graphs (diameter 2 graphs with order $\triangle^2+1$). (Such graphs exist only if $\triangle$ =2, 3, 7 and possibly 57). Thus Griggs and Yeh [11] conjectured that the best bound is $\triangle^2$ for any graph G with the maximum degree $\triangle\geq 2$. (This is not true for $\triangle=1$, For example, $\triangle(K_2)=1$ but $\triangle(K_2)=2$).

Graph products play an important role in connecting various useful networks and they also serve as natural tools for different concepts in many areas of research. In this paper, we have considered the graph formed by the homomorphic product of graphs and obtained a general upper bound for $L(2,1)$-labeling number in terms of the maximum degrees of the graphs. In the case of homomorphic product of graphs, the $L(2,1)$-labeling number of graph holds Griggs and Yeh's conjecture [11] with minor exception.

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2. A Labeling Algorithm

A subset X of $V(G)$ is called an i-stable set (or i-independent set) if the distance between any two vertices in X is greater than i. An 1-stable (independent) set is a usual independent set. A maximal 2-stable subset X of a set Y is a 2-stable subset of Y such that X is not a proper subset of any 2-stable subset of Y.

Chang and Kuo [9] proposed the following algorithm to obtain an $L(2,1)$-labeling and the maximum value of that labeling on a given graph.

Algorithm 2.1:

Input: A graph $G=(V,E)$

Output: The value k is the maximum label.

Idea: In each step i, find a maximal 2-stable set from the unlabeled vertices that are distance at least two away from those vertices labeled in the previous step. Then label all the vertices in that 2-stable set with the index i in the current stage. The label i starts from 0 and then increase by 1 in each step. The maximum label k is the final value of i.

Initialization: Set $X_{-1}=\phi; V=V(G); i=0$.

Iteration:

1. Determine Y_i and X_i.
   - $Y_i=\{u\in V: u$ is unlabelled and $d(u,v)\geq 2 \forall v\in X_{i-1}\}$.
   - Y_i is a maximal 2–stable subset of Y_i.
   - If $Y_i=\phi$ then set $X_i=\phi$.
2. Label the vertices of X_i (if there is any) with i.
3. $V\leftarrow V-X_i$.
4. If $V\neq\phi$, then $i\leftarrow i+1$, go to step 1.
5. Record the current i as k (which is the maximum label). Stop.

Thus k is an upper bound on $\lambda(G)$. Let u be a vertex with largest label k obtained by above algorithm. Set $I_1=\{i: 0\leq i\leq k-1$ and $d(u,v)=1$ for some $v\in X_i\}$.
$I_2=\{i: 0\leq i\leq k-1$ and $d(u,v)\leq 2$ for some $v\in X_i\}$.
$I_3=\{i: 0\leq i\leq k-1$ and $d(u,v)\geq 3$ for all $v\in X_i\}$.
Then Chang and Kuo showed that $\lambda(G)\leq k\leq |I_2|+|I_3|\leq |I_1|+|I_2|$.

In order to find k, it suffices to estimate $B=|I_1|+|I_2|$ in term of $\Delta(G)$. We will investigate the value B with respect to a particular graph (Homomorphic product of two graphs). The notations which have been introduced in this section will also be used in the following sections.

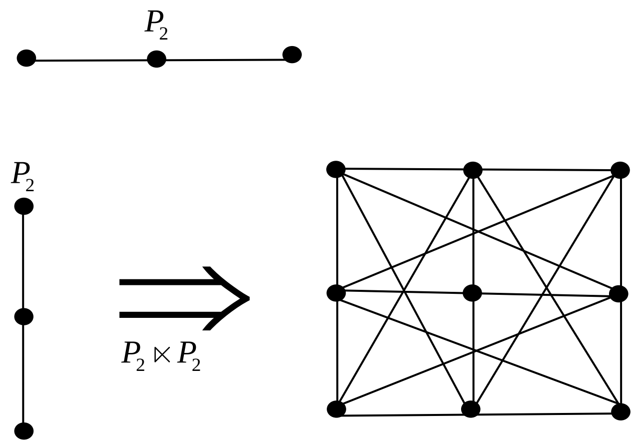

Fig.1. Homomorphic product $P_2\ltimes P_2$ of P_2 and P_2.

3. The Homomorphic Product of Two Graphs

The homomorphic product $G\ltimes H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$, in which the vertex (u,v) is adjacent to the vertex (u',v') if and only if either (a) $u=u'$ and v is adjacent to v' in H, or (b) $v=v'$ and u is adjacent u' in G, or (c) u is adjacent u' in G and v is not adjacent to v' in H, or (d) u is not adjacent u' in G and v is adjacent to v' in H. For example, we consider the Fig.1.

Lemma3.1. Let $x=(u,v)$ be a vertex of homomorphic product $G\ltimes H$ of G and H, where $u\in V(G)$, $u\in V(H)$. Let the number of vertices of G and H be η_G and η_H, respectively. Then degree of x is $\deg_{G\ltimes H}(x)=\deg_G(u)\eta_H+\deg_H(v)\eta_G-2\deg_G(u)\deg_H(v)$

Proof. For any graph G, we denote the set of edges of G by $E(G)$. By the definition of homomorphic product, the vertex (u,v) is adjacent to the vertex (u',v') if and only if either (a) $u=u'$ and $vv'\in E(H)$, or (b) $v=v'$ and $uu'\in E(G)$, or (c) $uu'\in E(G)$ and $vv'\notin E(H)$, or (d) $uu'\notin E(G)$ and $vv'\in E(H)$. Then from Fig.2, the number of vertices adjacent to $x=(u,v)$ in $G\ltimes H=$

$$\deg_{G\ltimes H}(x)=\deg_G(u)+\deg_H(v)+\deg_G(u)(\eta_H-\deg_H(v)-1)+\deg_H(v)(\eta_G-\deg_G(u)-1)$$
$$=\deg_G(u)\eta_H+\deg_H(v)\eta_G-2\deg_G(u)\deg_H(v)$$

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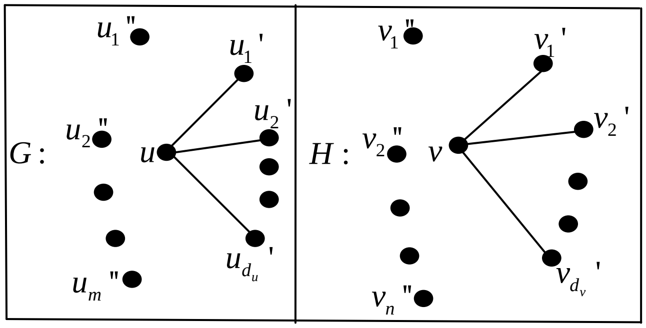

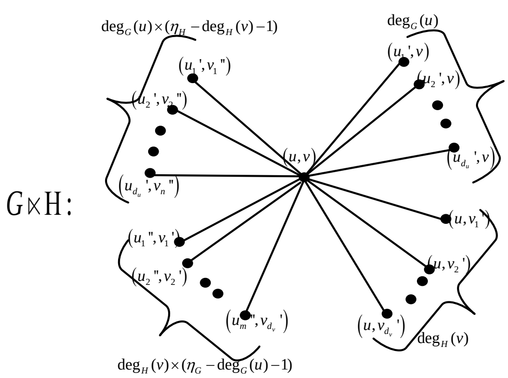

Fig.2. Degree of a vertex (u,v) in $G \ltimes H$

Lemma3.2. Let η_G and η_H denote the number of vertices of graphs G and H, respectively. Let the number of edges in G and H be E_G and E_H, respectively. If E denote the number of edges in homomorphic product $G \ltimes H$ of G and H, then $E = \eta_H^2 E_G + \eta_G^2 E_H - 4E_G E_H$.

Proof. Let $x=(u,v)$ be a vertex of $G \ltimes H$. Thus the degree of x is $\deg_{G\ltimes H}(x) = \deg_G(u)\eta_H + \deg_H(v)\eta_G - 2\deg_G(u)\deg_H(v)$. Then the number E of edges in $G \ltimes H$ can be find as $2E = \sum_{(u,v)\in G\ltimes H} \deg_{G\ltimes H}(u,v) =$

$\sum_{u\in G}\sum_{v\in H}\{\deg_G(u)\eta_H + \deg_H(v)\eta_G - 2\deg_G(u)\deg_H(v)\}$

$= \eta_H^2 \sum_{u\in G}\{\deg_G(u)\} + \eta_G^2 \sum_{v\in H}\{\deg_H(v)\}$

$-2\sum_{u\in G}\{\deg_G(u)\}\sum_{v\in H}\{\deg_H(v)\}$

$= \eta_H^2(2E_G) + \eta_G^2(2E_H) - 2(2E_G)(2E_H)$

Hence $E = \eta_H^2 E_G + \eta_G^2 E_H - 4E_G E_H$.

For any graph G, we denote the number of vertices of G by η_G. By the definition of the homomorphic product $G \ltimes H$ of graphs G and H, if $\eta_G = 1$ then $G \ltimes H = H$ and if $\eta_H = 1$ then $G \ltimes H = G$. Thus $\lambda(G \ltimes H) = \lambda(H)$ or $\lambda(G \ltimes H) = \lambda(G)$. Therefore in the following, we assume that $\eta_G \geq 2$ and $\eta_H \geq 2$.

4. $L(2,1)$-Labeling of the Homomorphic Product of Graphs

In this section, general upper bound for the $L(2,1)$-labeling number (λ-number) of homomorphic product $G \ltimes H$ in term of maximum degree of the graphs has been established. In this regard, we state and prove the following theorem.

Theorem4.1. Let Δ, Δ_G and Δ_H be the maximum degrees of $G \ltimes H$, G and H, respectively**.** Let η_G and η_H denote the number of vertices of G and H, respectively. Then $\lambda(G \ltimes H) \leq \Delta^2 - d_u d_v(\Delta_G + \Delta_H) - (\eta_G - d_u)d_u d_v - (\eta_H - d_v)d_u d_v$, where $d_{(u,v)} = \deg_{G\ltimes H}(u,v)$, $d_u = \deg_G(u)$, and $d_v = \deg_H(v)$ for any $u \in G, v \in H$.

Proof: Let $x=(u,v)$ be a vertex of $G \ltimes H$. Let $d = \deg_{G\ltimes H}(u,v)$, $d_u = \deg_G(u)$ and $d_v = \deg_H(v)$. Then $d = d_u\eta_H + d_v\eta_G - 2d_u d_v$ (by lemma3.1)

and $\Delta = \Delta(G \ltimes H) = \max_{u\in G, v\in H}\{d_u\eta_H + d_v\eta_G - 2d_u d_v\}$.

Let $N(x)$ be the set of neighbours of $x=(u,v)$. The value $d(\Delta-1)$ is the maximum possible number of vertices at distance two from vertex x in $G \ltimes H$. As we show below, this bound is not tight because (1) some vertices at distance 2 from vertex x will have common neighbours in $N(x)$, and (2) some edges of the graph $G \ltimes H$ have both end points in $N(x)$. To find a tighter bound for the maximum number of vertices which are at distance 2 from $x=(u,v)$ in $G \ltimes H$, we first find the number of vertices discuss in (1) and then the number of edges mentioned in (2).

Consider a vertex v' in H at distance 2 from v then there must be a path $v'v''v$ of length two between v' and v in H. Since the degree of the vertex u in G is d_u, then by the definition of homomorphic product $G \ltimes H$, there must be $2d_u + 1$ internally-disjoint paths (two paths are said to be internally-disjoint if they do not intersect each other) of length two between (u,v') and (u,v) in $G \ltimes H$, as shown in Fig.3. Since the vertex v has d_v neighbours v'' in H, and each neighbour v'' has at most $\Delta_H - 1$ neighbour vertices which are at distance 2 from v in H, then the maximum possible number of vertices (u,v') at distance 2 from (u,v) in $G \ltimes H$ is $d_v(\Delta_H - 1)$. As (u,v) has $d_v(2d_u+1)$ neighbours of the form (u,v'') or (u_i,v') or (u_i,v) in $G \ltimes H$, where u_i is adjacent to u in G and v'' is adjacent v in H, therefore we can assume that the maximum number of vertices (u,v') at distance 2 from (u,v) in $G \ltimes H$ can be as large as $d_v(2d_u+1)(\Delta_H - 1)$.

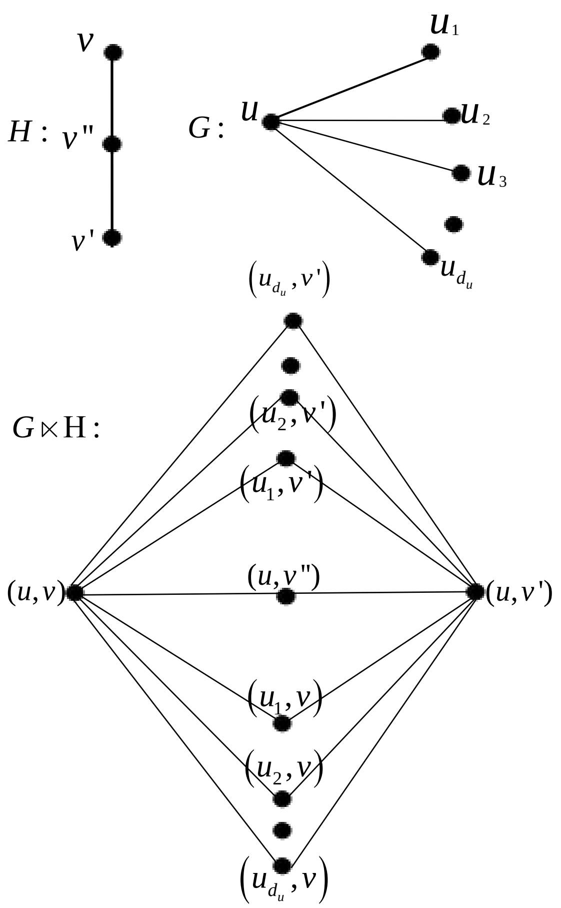

Fig.3 Paths between (u,v) and (u,v') in $G \ltimes H$

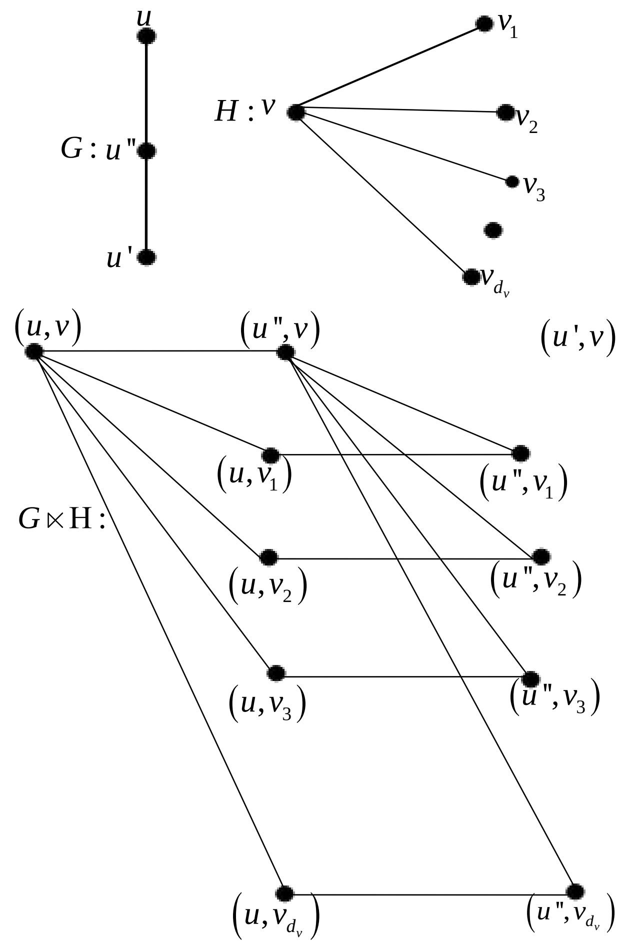

Fig.4. Paths between (u,v) and (u'',v_i) in $G \ltimes H$

Hence, a tighter bound for a maximum number of vertices at distance 2 from the vertex (u,v) in $G \ltimes H$ is $d(\Delta-1)-\{d_v(2d_u+1)(\Delta_H-1)-d_v(\Delta_H-1)\}$

$$= d(\Delta-1)-2d_ud_v(\Delta_H-1) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

In the similar way, if u' is a vertex at distance 2 from u in G, then there must be $(2d_v+1)$ internally disjoint paths of length 2 between (u,v') and (u,v) in $G \ltimes H$. Then the tighter bound (1) for the maximum number of vertices at distance 2 from the vertex (u,v) in $G \ltimes H$ reduces to $d(\Delta-1)-2d_ud_v(\Delta_H-1)-2d_ud_v(\Delta_G-1)$

$$= d(\Delta-1)-2d_ud_v(\Delta_G+\Delta_H-2) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Let $v_1,v_2,\ldots,v_{d_v}$ be the neighbours of v in H and u' be a vertex at distance 2 from u in G. By the definition of homomorphic product $G \ltimes H$, there must be two internally- disjoint paths of length 2 between (u,v) and every vertex (u'',v_i), $i=1,2,\ldots,d_v$ in $G \ltimes H$ as shown in Fig.4. Since the maximum number of vertices of the form u'' in G is d_u i.e the degree of the vertex u in G and there are d_v neighbours of v in H, then the maximum possible number of vertices of the form (u'',v_i) at distance 2 from (u,v) in $G \ltimes H$ is d_ud_v. In the similar way, if $u_1,u_2,\ldots,u_{d_u}$ be neighbours of u in G and $v'v''v$ be a path of length two in H, then the maximum possible number of vertices of the form (u_i,v''), $i=1,2,\ldots,d_u$ at distance 2 from (u,v) in $G \ltimes H$ is also d_ud_v. Therefore, an even tighter bound for the maximum number of vertices at distance 2 from the vertex (u,v) in $G \ltimes H$ is $d(\Delta-1)-2d_ud_v(\Delta_G+\Delta_H-2)-2d_ud_v$

$$= d(\Delta-1)-2d_ud_v(\Delta_G+\Delta_H-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

We can tighten this bound even more by considering the edges with both end points in $N(x)$. Let F be the subgraph induced in $G \ltimes H$ by $N(x)$. The edges of the subgraph F induced by the neighbours of $x=(u,v)$ can be divided into the following cases.

Case 1: Edges between (u',v) and (u_r,v'), where $u'u \in E(G)$ and $v'v \in E(H)$ and u_r is any vertex of G

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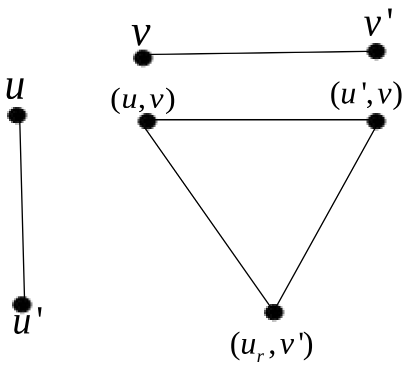

Fig.5

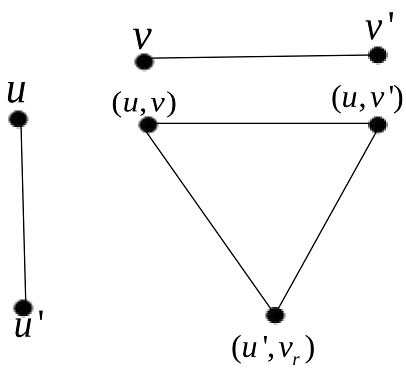

Fig.6

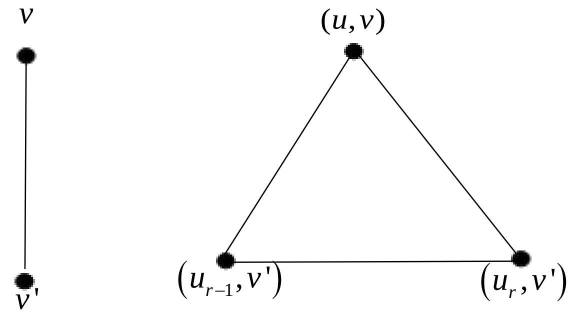

Fig.7

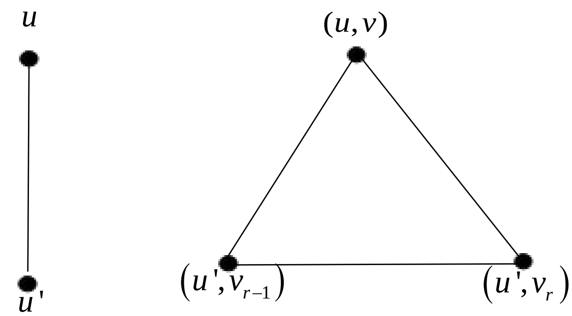

Fig.8

which is not adjacent to u and u' in G. For each neighbour vertex (u',v) of $x=(u,v)$ and any vertex (u_r,v'), (u_r,v') must be the common neighbour vertex of (u',v) and (u,v), then there must be an edge between (u_r,v') and (u',v) (see Fig.5). But there are at least $(\eta_G - d_u - \Delta_G)d_v$ neighbour vertices like (u_r,v') of $x=(u,v)$ and there are d_u neighbour vertices like (u',v) of $x=(u,v)$. Hence the number of these edges is at least $(\eta_G - d_u - \Delta_G)d_u d_v$.

Case 2: Edges between (u,v') and (u',v_r), where $u'u \in E(G)$ and $v'v \in E(H)$ and v_r is any vertex of H which is not adjacent to v and v' in H. For each neighbour vertex (u,v') of $x=(u,v)$ and any vertex (u',v_r), (u',v_r) must be the common neighbour vertex of (u,v') and (u,v), then there must be an edge between (u',v_r) and (u,v') (see Fig.6). But there are at least $(\eta_H - d_v - \Delta_H)d_u$ neighbour vertices like (u',v_r) of $x=(u,v)$ and there are d_v neighbour vertices like (u,v') of $x=(u,v)$. Hence the number of these edges is at least $(\eta_H - d_v - \Delta_H)d_u d_v$.

Case 3: Edges between (u_{r-1},v') and (u_r,v') where $u \in V(G)$ and $v'v \in E(H)$. For each neighbour vertices (u_r,v') and (u_{r-1},v') of (u,v) in $G \ltimes H$ (where u_r and u_{r-1} are any vertices of G which is neither equal to u nor adjacent to u and u_r is adjacent to u_{r-1} in G), there are $(\eta_G - d_u)d_v$ neighbour vertices like (u_r,v') of (u,v) in $G \ltimes H$. Hence the number of these edges is at least $(\eta_G - d_u)d_v$ (see Fig.7).

Case 4: Edges between (u',v_{r-1}) and (u',v_r) where $v \in V(H)$ and $u'u \in E(G)$. For each neighbour vertices (u',v_r) and (u',v_{r-1}) of (u,v) in $G \ltimes H$ (where v_r and v_{r-1} are any vertices of H which is neither equal to v nor adjacent to v and v_r is adjacent to v_{r-1} in H), there are $(\eta_H - d_v)d_u$ neighbour vertices like (u',v_r) of (u,v) in $G \ltimes H$. Hence the number of these edges is at least $(\eta_H - d_v)d_u$ (see Fig.8).

Case 5: Consider the Fig.9 for this case. If u' is adjacent to u in G and v' is adjacent to v in H, then (u,v) must be adjacent to both (u',v) and (u,v') in homomorphic product $G \ltimes H$ of graphs G and H. Hence the vertices of the subgraph F induced by the neighbours of $x=(u,v)$ should be all (u',v) and (u,v'). Since there are totally d_u neighbour vertices u' of u and totally d_v neighbour vertices v' of v, then the number of edges of the subgraph F induced by the neighbours of $x=(u,v)$ should be at least $d_u d_v$.

If ε be the number of edges of the subgraph F induced by the neighbours of $x=(u,v)$, then by the analysis of above cases

$$\varepsilon \geq (\eta_G - d_u - \Delta_G) d_u d_v + (\eta_H - d_v - \Delta_H) d_u d_v + (\eta_G - d_u) d_v + (\eta_H - d_v) d_u + 2 d_u d_v$$

$$\Rightarrow \varepsilon \geq (\eta_G - d_u - \Delta_G) d_u d_v + (\eta_H - d_v - \Delta_H) d_u d_v + \eta_G d_v + \eta_H d_u.$$

Whenever there is an edge in F, the number of vertices with distance 2 from $x=(u,v)$ will decrease by 2, hence the number of vertices with distance 2 from $x=(u,v)$ in $G \ltimes H$ will still need at least a decrease

$$(\eta_G - d_u - \Delta_G) d_u d_v + (\eta_H - d_v - \Delta_H) d_u d_v + \eta_G d_v + \eta_H d_u$$

from the value $d(\Delta-1) - 2 d_u d_v (\Delta_G + \Delta_H - 1)$. (The number $d(\Delta-1) - 2 d_u d_v (\Delta_G + \Delta_H - 1)$ is now the best tighter bound for the maximum number of vertices with distance 2 from $x=(u,v)$ in $G \ltimes H$).

Since the number of vertices with distance one from $x=(u,v)$ is d and the number of vertices with distance 2 from $x=(u,v)$ is no greater than $d(\Delta-1) - 2 d_u d_v (\Delta_G + \Delta_H - 1) - (\eta_G - d_u - \Delta_G) d_u d_v - (\eta_H - d_v - \Delta_H) d_u d_v - \eta_G d_v - \eta_H d_u$.

Then by the Labeling Algorithm $|I_1| \leq d$. $|I_2| \leq d + d(\Delta-1) - 2 d_u d_v (\Delta_G + \Delta_H - 1) - (\eta_G - d_u - \Delta_G) d_u d_v - (\eta_H - d_v - \Delta_H) d_u d_v - \eta_G d_v - \eta_H d_u$.

Therefore

$$\lambda(G \ltimes H) \leq k = |I_2| + |I_3| \leq |I_1| + |I_2| \leq d + d\Delta - 2 d_u d_v (\Delta_G + \Delta_H - 1) - (\eta_G - d_u - \Delta_G) d_u d_v - (\eta_H - d_v - \Delta_H) d_u d_v - \eta_G d_v - \eta_H d_u$$
$$= d\Delta + d_u \eta_H + d_v \eta_G - 2 d_u d_v - 2 d_u d_v (\Delta_G + \Delta_H - 1) - (\eta_G - d_u - \Delta_G) d_u d_v - (\eta_H - d_v - \Delta_H) d_u d_v - \eta_G d_v - \eta_H d_u$$
$$= d\Delta - d_u d_v (\Delta_G + \Delta_H) - (\eta_G - d_u) d_u d_v - (\eta_H - d_v) d_u d_v$$
$$\leq \max_{u \in G, v \in H} \{ d_u \eta_H + d_v \eta_G - 2 d_u d_v \} \Delta - d_u d_v (\Delta_G + \Delta_H) - (\eta_G - d_u) d_u d_v - (\eta_H - d_v) d_u d_v$$
$$= \Delta^2 - d_u d_v (\Delta_G + \Delta_H) - (\eta_G - d_u) d_u d_v - (\eta_H - d_v) d_u d_v.$$

Hence

$$\lambda(G \ltimes H) \leq k = |I_2| + |I_3| \leq |I_1| + |I_2| \leq \Delta^2 - d_u d_v (\Delta_G + \Delta_H) - (\eta_G - d_u) d_u d_v - (\eta_H - d_v) d_u d_v.$$

Corollary 4.2: Let Δ be the maximum degree of $G \ltimes H$, then $\lambda(G \ltimes H) \leq \Delta^2$ with some minor exceptions.

Proof: By theorem 4.1, $\lambda(G \ltimes H) \leq \Delta^2 - d_u d_v (\Delta_G + \Delta_H) - (\eta_G - d_u) d_u d_v - (\eta_H - d_v) d_u d_v$.

Define $f(s,t) = \Delta^2 - st(\Delta_G + \Delta_H) - (\eta_G - s)st - (\eta_H - t)st$.

Then $f(s,t)$ has the absolute maximum at (Δ_G, Δ_H) on $[0, \Delta_G] \times [0, \Delta_H]$.

$$f(\Delta_G, \Delta_H) = \Delta^2 - \Delta_G \Delta_H (\Delta_G + \Delta_H) - (\eta_G - \Delta_G) \Delta_G \Delta_H - (\eta_H - \Delta_H) \Delta_G \Delta_H = \Delta^2 - (\eta_G + \eta_H) \Delta_G \Delta_H.$$

Hence $\lambda(G \ltimes H) \leq \Delta^2 - (\eta_G + \eta_H) \Delta_G \Delta_H$.

If one of Δ_G or Δ_H is 1, then $G \ltimes H$ will be a general graph. Therefore taking, $\Delta_G \geq 2$ and $\Delta_H \geq 2$ (In this case we will clearly have $\eta_G \geq 3$ and $\eta_H \geq 3$). Thus

$$\Delta^2 - (\eta_G + \eta_H) \Delta_G \Delta_H \leq \Delta^2 - (3+3) 2.2 = \Delta^2 - 24 \leq \Delta^2.$$

Hence the upper bound $\lambda(G \ltimes H) \leq \Delta^2$ with some minor exceptions. Also note that generally the bound we obtain is much better than Griggs and Yeh conjectured.

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