Computation of Fifth Degree of Spline Function Model by Using C++ Programming

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Abstract

In this paper, a new quintic spline method developed for computing approximate solution of differential equations. It is shown that the present method is of the order three and four derivatives and gives approximations which are better. The numerical result obtained by the present method has been compared with the exact solution using C++ programming and also illustrate graphically the applicability of the new method. By getting the advantages of the mathematical building functions like pow (for power), exp (for exponential),…etc. are provided in C++ programming library, all processing steps are done efficiently and illustrated as Pseudocode model.

Keywords: - *Quintic spline, Differential equations, Building functions, Pseudocode.*

1. Introduction

A method for approximate solving initial value problems proposed for differential equations. In fact, this method is a variant of the well-known method of spline interpolation considered in [1]. A principal difference between considerations in [1] and ours is that, the new case of lacunary interpolations with others boundary conditions.

This method enables us to approximate the solution as well as its first and third derivatives at every point of the range of integration. We proved that this new method gives better numerical results than the previous known results. In recent years, Al-Said and Noor [2, 3], Khalifa and Noor [4] and Noor and Al-Said [5, 6] have used such types of penalty function in solving a class of contact problems in elasticity in conjunction with collocation, finite difference and spline techniques.

The general fourth order initial value problem considered is of the form

$$y^{(4)} + f(x)y = g(x)\,, \quad -\infty < x < \infty \tag{1}$$

With the boundary conditions

$$y(x_0) = y_0\ , y'(x_0) = y'_0\, , y''(x_0) = y''_0 \text{ and } y'''(x_0) = y'''_0 . \tag{2}$$

Where $x_0 = 0$, $x_n = 1$ and that $f \in C^{n-1}([0,1] \times R^4)$, and that f is Lipschitz continuous in y, y', y'', y''' and $y^{(4)}$, similarly for the third order initial value problems.

The aim of this paper is to construct a new spline method based on a quintic spline function that has a polynomial part and to develop numerical methods for obtaining smooth approximations for the solution of the problem (1) subject to the initial conditions (2).

The existence and uniqueness for spline function of degree five which interpolate the lacunary data (1, 3) is presented and examined in Section 2, we derive the numerical method and briefly discuss its error analysis theoretically in Section 3. Convergence analysis for second order, fourth order and fifth order methods is established in Section 4. Numerical results are presented to illustrate the applicability and accuracy their practical usefulness with C++ programming in Section 5. One of the C++ programming powerful includes (cmath) header file. The cmath header file provides a collection of functions that enables programmer to perform common mathematical calculations [7]. The instructions (codes) are illustrated in Pseudocode. Pseudocode is a compact and informal high-level description of a computer programming algorithm that uses the structural conventions of a programming language, but it is intended for human reading rather than machine reading [8].

2. Explanation of the Method

We consider a mesh with nodal points the x_j on $[a,b]$ such that;

$\Delta: \; a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ where $h = x_j - x_{j-1}, \; j = 0,1,2,\ldots,n$. Also we denote a quintic spline function $S_\Delta(x)$, interpolating to a function y(x) defined on [a, b] is such that:

$$S_i(x) = y_i + (x-x_i)y'_i + \frac{(x-x_i)^2}{2}a_{i,2} + \frac{(x-x_i)^3}{3!}y'''_i + (x-x_i)^4 a_{i,4} + (x-x_i)^5 a_{i,5} \quad (3)$$

$$S_i(x_{i+1}) = y_{i+1}, S'_i(x_{i+1}) = y'_{i+1} \text{ and } S'''_i(x_{i+1}) = y'''_{i+1} \quad (4)$$

On the last interval $[x_{n-1}, x_n]$ we define $S_{n-1}(x)$ as follows:

$$S_{n-1}(x) = y_{n-1} + (x-x_{n-1})y'_{n,} + \frac{(x-x_{n-1})^2}{2}a_{n-1,2} + \frac{(x-x_{n-1})^3}{3!}y_{n-1}^{(3)} + (x-x_{n-1})^4 a_{n-1,4} + (x-x_{n-1})^5 a_{n-1,5} \quad (5)$$

Where $a_{n-1,j}$, $j = 1, 3, 5 \text{ and } 6$, unknowns are to be determined.

Theorem 1: Existence and Uniqueness Spline Model

Gives the real numbers $y^{(r)}(x_i)$, *i*=0, 1, 2,…, *n* and *r= 0, 1, 3* and $y'(x_0)$ *and* $y'(x_n)$ then they exist a unique spline function of degree six from equations (3)-(5) such that:

$$\left.\begin{array}{l} S(x_i) = y(x_i) \\ S^{(r)}(x_i) = y^{(r)}(x_i), \; r = 1, 3 \\ \text{and} \\ S'(x_0) = y'(x_0) \; and \; S'(x_n) = y'(x_n) \end{array}\right\} \text{for } i = 0, 1, \ldots, n \quad (6)$$

Proof: For whole interval $[x_{i-1}, x_i]$ where $i = 0,1,2,\ldots,n$. Assuming y(x) to be the exact solution of the equation (3), obtained by the spline $S_i(x)$, along with the continuity condition of the first and third derivatives at $[x_{i-1}, x_i]$ in (4), the following consistency relations are derived:

$$h^2 a_{i,2} + h^4 a_{i,4} + h^5 a_{i,5} = y_{i+1} - y_i - hy'_i - \frac{h^3}{6} y'''_i$$

$$2ha_{i,2} + 4h^3 a_{i,4} + 5h^4 a_{i,5} = y'_{i+1} - y'_i - \frac{h^2}{2} y'''_i$$

$$24ha_{i,4} + 60h^2 a_{i,5} = y'''_{i+1} - y'''_i$$

Solving the above system, the coefficients of $S_i(x)$ on the interval $[x_i, x_{i+1}]$ for $i = 1,2,3,\ldots,n-2$.

$$a_{i,2} = \frac{5}{2h^2}(y_{i+1} - y_i) - \frac{1}{4h}(3y'_{i+1} + 7y'_i) + \frac{h}{48}(y'''_{i+1} - 3y'''_i) \quad (7)$$

$$a_{i,4} = -\frac{5}{2h^4}(y_{i+1} - y_i) + \frac{5}{4h^3}(y'_{i+1} + y'_i) - \frac{1}{48h}(3y'''_{i+1} + 7y'''_i) \quad (8)$$

$$a_{i,5} = \frac{1}{h^5}(y_{i+1} - y_i) - \frac{1}{2h^4}(y'_{i+1} + y'_i) + \frac{1}{24h^2}(y'''_{i+1} + y'''_i) \quad (9)$$

By solving these equations, we see that the coefficients $a_{n-1,i}$; i=2, 4 and 5 are uniquely determined, since we have three equations and three unknowns, and finally, we can find the coefficients of $S_{n-1}(x)$ similarly in the interval $[x_{n-1},x_n]$. Hence the proof is complete.

3. Convergence analysis

In this section, we investigate the convergence analysis of the quintic spline method described in Section 2.For this purpose, the error bound of the spline function $S(x)$ which is a solution of the problem (3) and (4) is obtained for the uniform partition I by the following theorem:

Theorem 2: Let $y \in C^6[0,1]$ is the exact solution of the differential equations (1) and $S(x)$ be a unique spline function of degree five which a solution of the problem (3) and (4). Then for $x \in [x_i, x_{i+1}]$; i=0,1, 2,…, n-1, we have

$$\left\| S_i^{(r)}(x) - y^{(r)}(x) \right\| \le \begin{cases} \frac{1}{120} h^{5-r} W_5(h) & for \quad r = 5 \\ \frac{1}{24} h^{5-r} W_5(h) & for \quad r = 4 \\ \frac{1}{6} h^{5-r} W_5(h) & for \quad r = 3 \\ \frac{1}{2} h^{5-r} W_5(h) & for \quad r = 2 \\ \frac{9}{4} h^{5-r} W_5(h) & for \quad r = 1 \\ \frac{7}{2} W_5(h) & for \quad r = 0 \end{cases}$$

where $W_6(h)$ denotes the modules of continuity of $y^{(5)}$, defined by $\|W_6(h)\| = \max\{|W_6(x)|; 0 \le x \le 1\}$

Proof:

Let $x \in [x_i, x_{i+1}]$ where i=1, 2, …, n-2.

From equation (4) and the Taylor's expansion formula, we have

$$S^{(5)}{}_i(x) = 120\ a_{i,5}$$

$$\left|S_i^{(5)}(x) - y^{(5)}(x)\right| = \left|\frac{120}{h^5} a_{i,5} - y^{(5)}(x)\right| = \left|120[hy_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \frac{h^4}{24} y_i^{(4)} + \frac{h^5}{120} y_i^{(5)}] - \frac{60}{h^4}[2y_i + hy_i'' + \frac{h^2}{2} y_i''' + \frac{h^3}{6} y_i^{(4)} + \frac{h^4}{24} y_i^{(5)}] + \frac{5}{h^2}[2y_i''' + hy_i^{(4)} + \frac{h^2}{2} y_i^{(5)}]\right|$$

$$\left|S_i^{(5)}(x) - y^{(5)}(x)\right| \le \frac{7}{2}\left|y_i^{(5)}(x) - y^{(5)}(x)\right| \le \frac{7}{2} W(h, f^{(5)})$$

And

$$S^{(4)}{}_i(x) = 24 a_{i,4} + 120(x - x_i)\ a_{i,5}$$

$$\left|S_i^{(4)}(x) - y^{(4)}(x)\right| = \left|24 a_{i,4} + 120(x - x_i)\ a_{i,5} - y^{(4)}(x)\right| \le \frac{7}{2}\left|y_i^{(5)}(x) - y^{(5)}(x)\right| \le \frac{9}{4} h W(h, f^{(5)})$$

And

$$S'''{}_i(x) = y_i''' + 24(x - x_i)\ a_{i,4} + 60(x - x_i)^2\ a_{i,5}$$

$$\left|S_i^{(3)}(x) - y^{(3)}(x)\right| \le \frac{h^2}{2}\left|y_i'''(x) - y'''(x)\right| \le \frac{h^2}{2} W(h, f^{(5)})$$

And

$$S_i''(x) = 2a_{i,2} + (x - x_i) y_i''' + 12(x - x_i)^2 a_{i,4} + 20(x - x_i)^3\ a_{i,5}$$

$$\left|S_i''(x) - y''(x)\right| \le \frac{h^3}{2} W(h, f^{(5)})$$

And

$$S_i'(x) = y_i' + 2(x - x_i) a_{i,2} + \frac{(x - x_i)^2}{2!} y_i''' + 4(x - x_i)^3 a_{i,4} + 5(x - x_i)^4\ a_{i,5}$$

$$\left|S_i'(x) - y'(x)\right| \le \frac{h^4}{24} W(h, f^{(5)})$$

And

$$S_i(x) = y_i + (x - x_i) y_i' + \frac{(x - x_i)^2}{2} a_{i,2} + \frac{(x - x_i)^3}{3!} y_i''' + (x - x_i)^4 a_{i,4} + (x - x_i)^5\ a_{i,5}$$

$$\left|S_i(x) - y(x)\right| \le \frac{h^5}{120} W(h, f^{(5)})$$

This proves Theorem 2 for $x \in [x_{i-1}, x_i]$, similarly we can obtain the result for $x \in [x_{n-1}, x_n]$. This completes the proof.

3. Illustration Examples:

This section, several numerical examples are given to illustrate the properties of the method and all of them were performed on the computer using a program written in [7, 8]. The absolute errors in Tables 1–2 are the values of $|\mathrm{y(x)} - S(x)|$ at selected points, and also the following figures are shown that if increases of the order derivatives increases the errors.

Problem 1: we consider the initial value problem $y^{(5)} = y$ where $x \in [0,1]$,

$y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = 0,$ clearly that, the exact solution is

$y(x) = e^x$.

The Pseudocode of problem 1 is:

for (**i** = start point **to** end point , increase start point by **h** for each step)

{

Step 1: Find

$y_1 = y_0$+(h* $y_0^{'}$)+((pow(h,2)/2)* $y_0^{''}$)+((pow(h,3)/6)* $y'''(0)$)+((pow(h,4)/24)* $y^{(4)}(0)$)+((pow(h,5)/120)* $y^{(5)}(0)$)

$y_1^{'} = y_0^{'}$+(h* $y_0^{''}$)+((pow(h,2)/2)* $y'''(0)$)+((pow(h,3)/6)* $y^{(4)}(0)$)+((pow(h,4)/24)* $y^{(5)}(0)$)

y_1''' = $y'''(0)$ +(h* $y^{(4)}(0)$)+((pow(h,2)/2)* $y^{(5)}(0)$)

Step 2: Find

S'' =(5/pow(h,2))*($y_1 + y_0$)+(1/(2*h))*(7* $y_1^{'}$+(3* $y_0^{'}$))+(h/24)*(3* y_1''' -25* $y'''(0)$)

$S^{(4)}$=(60/pow(h,4))*(y_1 - y_0)-(30/pow(h,3))*($y_1^{'}$+ $y_0^{'}$)+(1/(2*h))*(7* y_1''' + 3* $y'''(0)$)

$S^{(5)}$=(120/pow(h,5))*(y_1 - y_0)-(60/pow(h,4))*($y_1^{'}$+ $y_0^{'}$)+(5/pow(h,2))*(y_1''' + $y'''(0)$)

Step 3: Find

y'' =(0.5)*((exp(h)+exp(-h)))

$y^{(4)}$ = y''

$y^{(5)}$ = (0.5)*(exp(h)- exp(-h))

Step 4: Find

Error 2 = S'' - y''

Error 4 = $S^{(4)}$ - $y^{(4)}$

Error 5 = $S^{(5)}$ - $y^{(5)}$

Step 5: Print Error 1 , Error 2 , Error3 respectively.

}

Problem 2: Consider that the fifth order boundary value problem $y^{(5)} - y^{(4)} - y' + y = 0$ where $x \in [0,1]$, $y(0) = y'(0) = y'''(0) = y^{(4)}(0) = 0$ and $y''(0) = 1$ the exact solution is $y(x) = \frac{1}{4}e^{-x} + \frac{1}{4}e^{x} - \frac{1}{2}\cos(x)$

The Pseudocode of problem 2 is:

for (**i** = start point **to** end point , increase start point by **h** for each step)

{

Step 1: Find

$y_1 = y_0$+(h* y_0')+((pow(h,2)/2)* y_0'')+((pow(h, 3)/6)* $y'''(0)$)+((pow(h,4)/24)* $y^{(4)}(0)$)+ ((pow(h,5)/120) * $y^{(5)}(0)$)

$y_1' = y_0'$+(h* y_0'')+((pow(h,2)/2)* $y'''(0)$)+((pow (h,3)/6)* $y^{(4)}(0)$)+((pow(h,4)/24)* $y^{(5)}(0)$)

y_1''' = $y'''(0)$ + (h * $y^{(4)}(0)$) + ((pow(h,2) / 2) * $y^{(5)}(0)$)

Step 2: Find

S''=(5/pow(h,2))*(- $y_1 + y_0$) +(1/(2*h))*(7* y_1'+(3* y_0'))+(h/24)*(3* y_1''' - 25* $y'''(0)$)

$S^{(4)}$=(60/pow(h,4))*($y_1 - y_0$)- (30/pow(h,3))*($y_1' + y_0'$) +(1/(2*h))*(7* y_1''' +3* $y'''(0)$)

$S^{(5)}$=(120/pow(h,5))*($y_1 - y_0$)- (60/pow(h,4))*($y_1' + y_0'$)+(5/pow(h,2))*($y_1''' + y'''(0)$)

Step 3: Find

y'' = exp(-h))

$y^{(4)}$ = y''

$y^{(5)}$ = -(exp(-h))

Step 4: Find

Error 2 = S'' - y''

Error 4 = $S^{(4)}$ - $y^{(4)}$

Error 5 = $S^{(5)}$ - $y^{(5)}$

Step 5: Print Error 1 , Error 2 , Error3 respectively.

}

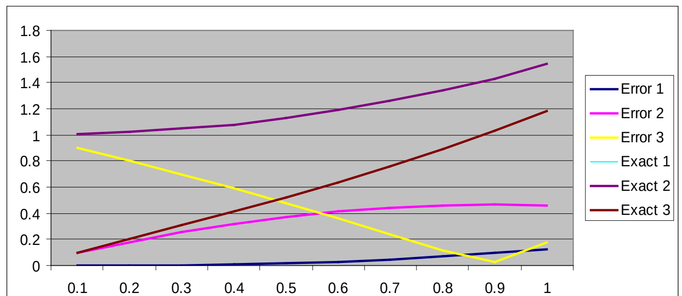
Table (1): Maximum errors in solution of problem 1

h	$E^{(2)}$	$E^{(4)}$	$E^{(5)}$
0.1	$1.62 * 10^{-4}$	$9.49 * 10^{-2}$	$9.01 * 10^{-1}$
0.05	$2.06 * 10^{-5}$	$4.85 * 10^{-2}$	$9.59 * 10^{-1}$
0.01	$1.58 * 10^{-7}$	$3.69 * 10^{-2}$	$2.65 * 10^{0}$

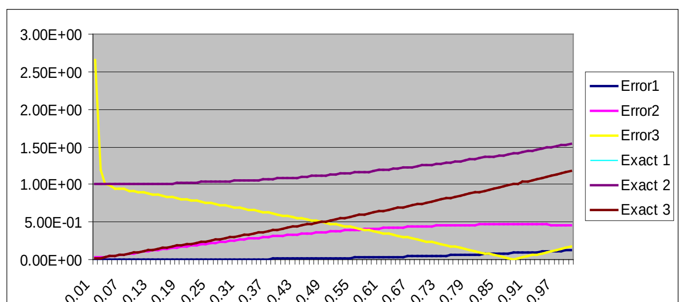
Table (2): Maximum errors in solution of problem 2

h	$E^{(2)}$	$E^{(4)}$	$E^{(5)}$
0.1	$9.9996 * 10^{-2}$	$4.8766 * 10^{-3}$	$9.3849 * 10^{-2}$
0.05	$5 * 10^{-2}$	$1.409 * 10^{-3}$	$3.8843 * 10^{-2}$
0.01	$1 * 10^{-2}$	$2.6651 * 10^{-2}$	$16.411 * 10^{-1}$

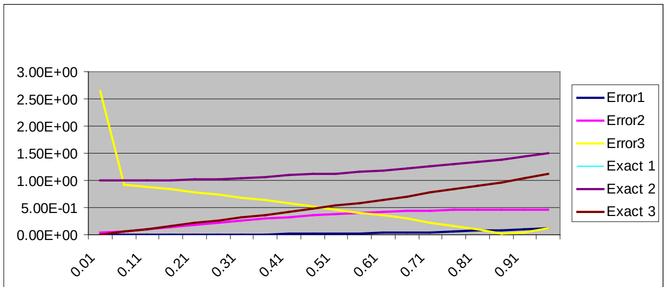
The following figures observe the numerical results with respect two orders of derivative:



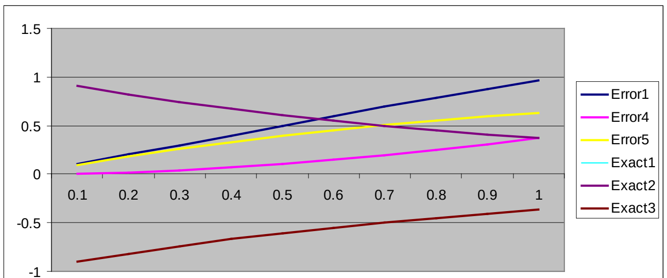
Absolute Error Graph for problem 1, when h = 0.1



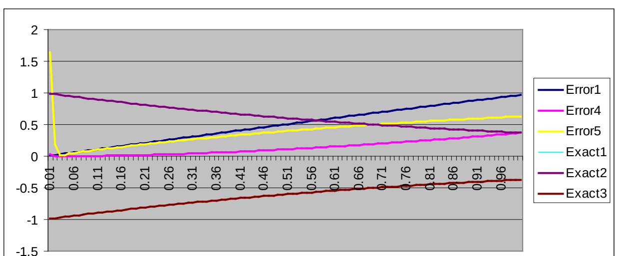
Absolute Error Graph for problem 1, when h = 0.01



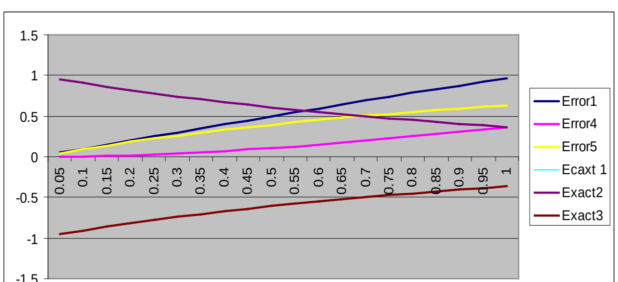
Absolute Error Graph for problem 1, when h = 0.05



Absolute Error Graph for problem 2, when h = 0.1



Absolute Error Graph for problem 2, when h = 0.01



Absolute Error Graph for problem 2, when h = 0.05

5. Discussion:

A new technique, using the Taylor series, to numerically solution the pantograph equations is presented. It is observed that the method has the best advantage when the known functions in equation can be expanded to Taylor series with converge rapidly. In order to get the best approximation, we take more terms from the Taylor expansion of functions; that is, the truncation limit N must be chosen large enough.

On the other hand, from Table 1, it may be observed that the solutions found for different h show close agreement for various values of x. In particular, our results in tables are usually better than the other methods, are shown in the above figures. Another considerable advantage of the method is that Taylor coefficients of the solution are found very easily by using the computer programs.

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