Regular Maximum Principle in the Time Optimal Control Problems with Time-Varying State Constraint

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Abstract

In this paper the necessary conditions for optimality are obtained for regular solutions of the time optimal control problems with time-varying state constraint. Is constructed an example of time optimal control problem with active time varying state constraint, where the optimal trajectory achieves the boundary of the state constraint infinitely number times, the adjoint function is absolutely continuous, non trivial and at the same time the adjoint system is homogeneous.

***Keywords**: Optimal Control, State Constraints, Similar Solutions, Maximum Principle*

Introduction

For the first time Pontryagin maximum principle for problems with state constraints was obtained by Gamkrelidze R. V. in 1959 [1], [2]. In 1963 another variant of the maximum principle [3] has been received. After that, the matter was the subject of many studies [5], [6]. This list of works is not exhaustive.

In the case, where there is a restriction only on the control function and there are no state constraints, necessary optimality conditions gives Pontryagin's maximum principle [1]. These problems have been well studied because of the absolutely continuity and non-triviality of adjoint functions.

The optimal control problems with state constraints are recognized as an important and difficult class of the similar problems, since the maximum principle for such problems [3], [5], [6] contains an unknown infinite-dimensional Lagrange multiplier of the complex nature-bounded regular Borel measure which has a rather complicated relationship with the optimal trajectory. Therefore, the optimal control problem with state constraints are outside the scope of the effective application of the Pontryagin maximum principle [1]. Questions arise: Are there any solutions of an optimal control problem for which the corresponding conjugate function is non-trivial and absolutely continuous, and if so, how to find them?

Applying a similar technique in [14], [15] we try to answer these questions for non-autonomous systems with time-varying state constraint.

Analogical question about the structure of the measures appearing in the ratios of the maximum principle for the classical optimal control problem was considered by W. W. Hager [7], K. Malanowski [8], Hoàng Xuàn Phù [9], H.Maurer [10], A. A. Milutin [11], J. F. Bonnans [13], for the differential inclusions by S. M. Aseev [12]. In [7] -[11], [13] sufficient conditions for the absence of a singular component obtained under the condition that the time optimal control function is continuous and takes values strictly in the interior of U.

In [12], sufficient conditions for the absence of a singular component obtained under the condition that the set of admissible velocities are strictly convex and the Hamiltonian of the system satisfies certain smoothness conditions.

These and the results, obtained in this study are difficult to compare: they are all proved under different assumptions and have different conditions. Apparently, this issue need a separate study.

Statement of the problem

Consider the time optimal control problem with state constraint for non-autonomous system

$$\begin{aligned} &\dot{x} = f(x,u,t), \\ &x(0) = x_\alpha, \quad x(T) = x_\beta, \\ &u(t) \in U(t), \quad a.e. \quad t \in [0,T], \\ &g(x(t),t) \le 0, \quad \forall t \in [0,T] \\ &\quad T \to \min \end{aligned} \qquad (1)$$

Here, $x \in E^n$ –state variable, $u \in E^m$ -control parameter. Let $\Omega(E^n)$ be the set of all nonempty compact and $conv\Omega(E^n)$ the set of all nonempty compact convex subsets of E^n. Functions $f(x,u,t), \frac{\partial f}{\partial x}$ are continuous in (x,u) and measurable in t. Let the set valued map $U : E^1 \to \Omega(E^m)$ be measurable and satisfy the estimate $|U(t)| \le k(t)$, where $k(t)$ is a scalar function, Lebesgue integrable on any finite time interval $[0,T]$. $f(x,U(t),t) \in conv\Omega(E^n),\ t \in [0,T].$ $g(x,t)$ is continuously differentiable with respect to the set of variables, and $\frac{\partial g(x,t)}{\partial x} \ne 0,$ where $g(x,t) = 0.$

Let $H(F,\psi) = \max\{(f,\psi) : f \in F\}$ be the support function of the set $F \subset E^n$ in the direction of ψ, where (f,ψ) denotes the scalar product of vectors f and ψ. $T(A,a)$ and $N(A,a)$ are the tangent and normal cones to a closed, convex set A at a point $a \in A$, respectively. All finite-dimensional vectors are considered column vectors.

Function $u(t) \in U(t),\ t \in [0,T]$ is called an admissible control on the interval $[0,T]$, if it is measurable and one–valued branch of the multivalued mapping $U(t)$, so that the corresponding solution $x(t),\ t \in [0,T]$ of the given system of differential equations satisfies the initial condition $x(0) = x_\alpha$ and the inequality $g(x(t),t) \le 0,\ \forall t \in [0,T]$. The challenge is in finding an admissible control $u(t), t \in [0,T]$, such that the corresponding trajectory $x(t),\ t \in [0,T]$ satisfies condition $x(T) = x_\beta$ and T is minimal.

This work is dedicated to deriving the maximum principle for which the optimal solution is regular.

Optimal solution is called regular, if the corresponding conjugate function is a nontrivial absolutely continuous function.

Lemma 1. Let $x(t),\ t \in [0,T]$ be the some absolutely continious function, for which $g(x(t),t) \le 0,\ t \in [0,T]$. Then for almost all $t \in \{s \in [0,T] : g(x(s),s) = 0\}$ the inequality

$$\frac{\partial g(x(t),t)}{\partial x}\dot{x}(t) + \frac{\partial g(x(t),t)}{\partial t} \le 0$$

is true.

Proof. Let there exists the derivative $\dot{x}(t_0)$ at the point $t_0 \in \{s \in [0,T] : g(x(s),s) = 0\}$ and $t_0 \ne 0,\ t_0 \ne T$.

By the definition

$$\frac{dg(x(t_0),t_0)}{dt} = \lim_{\Delta t \to 0} \frac{g(x(t_0+\Delta t), t_0+\Delta t) - g(x(t_0),t_0)}{\Delta t},$$

and for the sufficiently small $\Delta t > 0$

$$\frac{g(x(t_0+\Delta t), t_0+\Delta t) - g(x(t_0),t_0)}{\Delta t} \le 0,$$

because $dg(x(t_0),t_0) = 0$ and $g(x(t_0+\Delta t), t_0+\Delta t) \le 0.$

On the other hand

$$\frac{g(x(t_0+\Delta t), t_0+\Delta t) - g(x(t_0),t_0)}{\Delta t} =$$

$$= \frac{g(x(t_0+\Delta t), t_0+\Delta t) - g(x(t_0),t_0+\Delta t)}{\Delta t} + \frac{g(x(t_0), t_0+\Delta t) - g(x(t_0),t_0)}{\Delta t} =$$

$$= \frac{g(x(t_0+\Delta t), t_0+\Delta t) - g(x(t_0),t_0+\Delta t)}{\Delta t} + \frac{g(x(t_0), t_0+\Delta t) - g(x(t_0),t_0)}{\Delta t} =$$

$$= \frac{g(x(t_0+\Delta t), t_0+\Delta t) - g(x(t_0),t_0+\Delta t)}{x(t_0+\Delta t) - x(t_0)} \times \frac{x(t_0+\Delta t) - x(t_0)}{\Delta t} + \frac{g(x(t_0), t_0+\Delta t) - g(x(t_0),t_0)}{\Delta t} =$$

$$= \left(g_x(x(t_0), t_0+\Delta t) + \frac{o(\|x(t_0+\Delta t) - x(t_0)\|)}{x(t_0+\Delta t) - x(t_0)} \right) \times \frac{x(t_0+\Delta t) - x(t_0)}{\Delta t} + g_t(x(t_0),t_0) + \frac{o(|\Delta t|)}{\Delta t} \le 0$$

for the sufficiently small $\Delta t > 0$.

We have

$$\lim_{\Delta t \to 0}\left[\left(g_x(x(t_0), t_0+\Delta t) + \frac{o(\|x(t_0+\Delta t) - x(t_0)\|)}{x(t_0+\Delta t) - x(t_0)} \right) \times \frac{x(t_0+\Delta t) - x(t_0)}{\Delta t} + g_t(x(t_0),t_0) + + \frac{o(|\Delta t|)}{\Delta t}\right] =$$

$$= \frac{\partial g(x(t_0),t_0)}{\partial x} \cdot \dot{x}(t_0) + \frac{\partial g(x(t_0),t_0)}{\partial t} \le 0$$

At the end points 0 and T held a similar argument regarding the left (at T) and the right (at 0) limits in the definition of the derivative.

From the arbitrariness of $t \in \{s \in [0,T] : g(x(s),s) = 0\}$ and the absolutely continuity of $x(t),\ t \in [0,T]$ we conclude that

$$\frac{\partial g(x(t),t)}{\partial x}\dot{x}(t)+\frac{\partial g(x(t),t)}{\partial t}\leq 0,$$
$$a.e.\quad t\in\{s\in[0,T]:g(x(s),s)=0\}$$

Thus, we have proved the lemma.
From the lemma 1 we have the

Lemma 2. If $x(t),\ t\in[0,T]$ is a some admissible trajectory of (1), then

$$U(x(t),t)\neq\varnothing,$$

where

$$U(x(t),t)=\begin{cases} u\in U(t): g_x(x(t),t)\cdot f(x(t),u,t)+\\ +g_t(x(t),t)\leq 0,\\ if\quad a.e.\quad t\in\{s\in[0,T]:\\ g(x(s),s)=0\},\\ U(t),\ if\quad a.e.\\ t\in[0,T]\setminus\{s\in[0,T]:\\ g(x(s),s)=0\}.\end{cases}$$

The proof of the Lemma follows immediately from the lemma 1.

Let $(\overline{x}(t),\overline{u}(t)),\ t\in[0,\overline{T}]$ be a solution to (1). Consider the corresponding auxiliary

$$\begin{aligned}&\dot{x}=f(x,u,t),\quad a.e.\quad t\in[0,T],\\ &x(0)=\hat{x}_\alpha,\quad x(T)=\hat{x}_\beta,\\ &u(t)\in U(t),\ a.e.\quad t\in[0,T],\\ &\qquad T\to\min,\end{aligned}\tag{2}$$

and the intermediate

$$\begin{aligned}&\dot{x}=f(x,u,t),\\ &x(0)=x_\alpha,\quad x(T)=x_\beta,\\ &u\in U(\overline{x}(t),t),\ a.e.\quad t\in[0,T],\\ &g(x(t),t)\leq 0,\quad \forall t\in[0,T],\\ &\qquad T\to\min,\end{aligned}\tag{3}$$

problems.

Definition. *If there are the points* $\hat{x}_\alpha.\hat{x}_\beta$ *and the decision* $(x_0(t),u_0(t)),\ t\in[0,\overline{T}]$ *of the problem (2), for which the inclusion*

$$T(f(\overline{x}(t),U(\overline{x}(t),t),t),\dot{\overline{x}}(t))\subset T(f(x_0(t),$$
$$U(t),t),\dot{x}_0(t)),\ a.e.\quad t\in[0,\overline{T}],$$

holds, then $(x_0(t),u_0(t)),\ t\in[0,\overline{T}]$ *is called similar to the solution* $(\overline{x}(t),\overline{u}(t)),\ t\in[0,\overline{T}]$ *of (1).*

Note. The main requirement of this definition is the equality of optimal values of quality criteria in the original (problem with state constraints (1)) and auxiliary (the problem without phase constraints (2)) problems.

Theorem 1. *Suppose, there exists a similar solution* $(x_0(t),u_0(t)),\ t\in[0,\overline{T}]$ *of (2) to* $(\overline{x}(t),\overline{u}(t)),\ t\in[0,\overline{T}]$. *Then, there exists a nonzero absolutely continuous solution of the adjoint system of differential equations*

$$\dot{\psi}(t)=-\frac{\partial f^*(x_0(t),u_0(t),t)}{\partial x}\psi(t),\quad a.e.\quad t\in[0,\overline{T}],$$

for which, the maximum condition

$$\max\{(f(\overline{x}(t),u,t),\psi(t):u\in U(\overline{x}(t),t)\}=$$
$$(\dot{\overline{x}}(t),\psi(t)),\ a.e.\quad t\in[0,\overline{T}]$$

holds.
If the additional condition

$$\left(\frac{\partial f^*(\overline{x}(t),\overline{u}(t),t)}{\partial x}-\frac{\partial f^*(x_0(t),u_0(t),t)}{\partial x}\right)\psi(t)=0,$$
$$a.e.\ t\in[0,\overline{T}]$$

holds, then the conjugate system of differential equations will have the form

$$\dot{\psi}(t)=-\frac{\partial f^*(\overline{x}(t),\overline{u}(t),t)}{\partial x}\psi(t),\quad a.e.\quad t\in[0,\overline{T}].$$

Proof. At first consider the intermediate optimal control problem (3). We denote $X(t)=\{x\in E^n:g(x,t)\leq 0\},\ t\in[0,T]$ and $T(X(t),x)=\{z\in E^n:g_x(x,t)z+g_t(x,t)\leq 0\}$, $x\in X(t),\ t\in[0,T]$. It can be easily shown that the solution $(\overline{x}(t),\overline{u}(t)),\ t\in[0,\overline{T}]$ of (1) is the solution of the problem (3) also. On the other hand, it is satisfied the following inclusion

$$f(\overline{x}(t),U(\overline{x}(t),t),t)\subset T(X(t),\overline{x}(t)),\quad a.e.\quad t\in[0,\overline{T}].$$

In other words the tangent cone $T(X(t),\overline{x}(t)),\ t\in[0,\overline{T}]$ to the $X(t)$ at the point $\overline{x}(t)\in X(t)$ includes the set of all admissible velocities of the problem (3).

Very important question arises: Is the state constraint $x(t) \in X(t),\ t \in [0,T]$ aktive? It turns out, there are cases, when the state constraint $x(t) \in X(t),\ t \in [0,T]$ is inactive in the problem (3). Namely, in the case, when there exists the similar solution of the auxiliary problem (2) to the solution $(\bar{x}(t), \bar{u}(t),\ t \in [0,\bar{T}]$ of the problem (3), the state constraint $x(t) \in X(t),\ t \in [0,T]$ is inactive, that is the following considerations are true.

Because of $(x_0(t), u_0(t)),\ t \in [0,\bar{T}]$ is a solution to problem (2), which is a problem without phase constraints, we can apply the Pontryagin maximum principle for this decision [1]. In other words, there exists a nontrivial absolutely continuous function $\psi(t),\ t \in [0,\bar{T}],$ as a solution of the dual system of equations

$$\dot{\psi}(t) = -\frac{\partial f^*(x_0(t), u_0(t), t)}{\partial x}\psi(t), \quad a.e.\ \ t \in [0,\bar{T}],$$

for which, the maximum condition

$$\max\{(f(x_0(t), u, t), \psi(t) : u \in U(t))\} = $$
$$= (\dot{x}_0(t), \psi(t)),\ \ a.e.\ \ t \in [0,\bar{T}]$$

holds.

The last equality means that

$$\psi(t) \in N(f(x_0(t), U(t), t), \dot{x}_0(t)), a.e. t \in [0,\bar{T}] \quad (4)$$

where $N(A,a)$ is a normal cone of the set $A \in conv\Omega(E^n)$ at a point $a \in A$.

By the definition of the similar solutions and conditions of the theorem,

$$T(f(\bar{x}(t), U(\bar{x}(t), t), t), \dot{\bar{x}}(t)) \subset$$
$$\subset T(f(x_0(t), U(t), t), \dot{x}_0(t)),\ \ a.e.\ \ t \in [0,\bar{T}],$$

therefore,

$$N(f(x_0(t), U(t), t), x_0(t)) \subset$$
$$\subset N(f(\bar{x}(t), U(\bar{x}(t), t), t)\dot{\bar{x}}_0(t)),\ \ a.e.\ \ t \in [0,\bar{T}],$$

From the inclusion (4), we conclude that the absolutely continuous nontrivial one valued branch $\psi(t),\ t \in [0,\bar{T}]$ of the multivalued map $N(f(x_0(t), U(t), t), \dot{x}_0(t))$ is a one valued branch of the multivalued mapping

$$N(f(\bar{x}(t), U(\bar{x}(t), t), t)\dot{\bar{x}}(t)), \quad t \in [0,\bar{T}]$$

also.

This means that

$$\max\{(f(\bar{x}(t), u, t), \psi(t)) : u \in U(\bar{x}(t), t)\} =$$
$$= (\dot{\bar{x}}(t), \psi(t)),\ \ a.e.\ \ t \in [0,\bar{T}]$$

In case of the additional condition of the theorem, it is easy to show that the conjugate function $\psi(t),\ t \in [0,\bar{T}]$ satisfies the system of the differential equations

$$\dot{\psi}(t) = -\frac{\partial f^*(\bar{x}(t), \bar{u}(t), t)}{\partial x}\psi(t), \quad a.e.\ \ t \in [0,\bar{T}],$$

The theorem is proved.

Consider a linear optimal control problem with state constraint

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \eta(u,t), \\ x(0) \in x_\alpha,\ x(T) \in x_\beta, \\ u(t) \in U(t),\ \ a.e.\ \ t \in [0,T], \\ g(x,t) \le 0,\ \ \forall t \in [0,T] \\ T \to \min \end{cases} \quad (5)$$

and the corresponding problem without state constraint

$$\begin{cases} x(t) = A(t)x(t) + \eta(u,t), \\ x(0) \in \hat{x}_\alpha,\ \ x(T) \in \hat{x}_\beta, \\ u(t) \in U(t),\ \ a.e.\ \ t \in [0,T], \\ T \to \min, \end{cases} \quad (6)$$

Here, A is a given $n \times n$ matrix of bounded measurable elements, $\eta(u,t)$ measurable in t and continuous in u, $\eta(U(t),t) \in conv\Omega(E^n),\ t \in [0,T]$.

Consequence. *Let there exist a similar solution* $(x_0(t), u_0(t)),\ t \in [0,\bar{T}]$ *of (6) to the solution* $(\bar{x}(t), \bar{u}(t),\ t \in [0,\bar{T}]$ *of (5). Then there exists a nontrivial solution of the adjoint system of differential equations*

$$\dot{\psi}(t) = -A^*(t)\psi(t),\ \ a.e.\ \ t \in [0,\bar{T}],$$

for which, the maximum condition

$$\max\{(\eta(u,t), \psi(t), u \in U(\bar{x}(t), t)\} =$$
$$= (\eta(\bar{u}(t), t), \psi(t)),\ \ a.e.\ \ t \in [0,\bar{T}]$$

holds.

The proof follows easily from the proof of the theorem.

Note. For linear on x systems, an additional condition is always satisfied, since in this case

$$\frac{\partial f^*(\bar{x}(t), \bar{u}(t), t)}{\partial x} = A^*(t) = \frac{\partial f^*(x_0(t), u_0(t), t)}{\partial x},$$
$$a.e.\ \ t \in [0,\bar{T}],$$

Thus, in cases, linear with respect to the phase coordinates, we need only condition for the existence of the similar solutions.

Example

Consider the linear time optimal control problem

$$\begin{cases}\dot{x}_1 = x_2, \\ \dot{x}_2 = u,\end{cases} \quad x_0 = (0,0), \quad x(T) = (1,0),$$

$$|u| \le 1, \quad x(t) \in X(t), \quad t \in [0,T],$$

where

$$X(t) = \begin{cases} (x_1, x_2) \in E^2 : \ 0 \le x_2 \le t + T_{n0}, \\ if \quad t \in \left[T_{n0}, T_{n0} + \frac{1}{2^{n/2}} \right], \\ (x_1, x_2) \in E^2 : \ 0 \le x_2 \le T_{n1} - t, \\ if \quad t \in \left[T_{n0} + \frac{1}{2^{n/2}}, T_{n1} \right], \end{cases}$$

and

$$T_{n0} = \sum_{i=0}^{n-1} \frac{2}{2^{\frac{i}{2}}} - 2, \quad T_{n1} = \sum_{i=0}^{n} \frac{2}{2^{\frac{i}{2}}} - 2.$$

$$T \to \min.$$

Solution.

A straightforward calculation shows that for T_{n0} and T_{n1} the following relations

$$T_{n0} = \frac{2^{\frac{n+2}{2}} - 2^{\frac{3}{2}}}{2^{\frac{n+1}{2}} - 2^{\frac{n}{2}}}, \qquad T_{n1} = \frac{2^{\frac{n+3}{2}} - 2^{\frac{3}{2}}}{2^{\frac{n+2}{2}} - 2^{\frac{n+1}{2}}}$$

hold.

For each time interval $T_n, \quad n = 1,2,...$ on the first half $\left[T_{n0}, T_{n0} + \frac{1}{2^{n/2}} \right]$ the function $g(x,t)$ is $g(x,t) = x_2 - t - T_{n0}$. On the second half $\left[T_{n0} + \frac{1}{2^{n/2}}, T_{n1} \right]$ the function $g(x,t)$ is $g(x,t) = x_2 - T_{n1} + t.$

Consequently,

$$g_x(\bar{x}(t),t) f(\bar{x}(t),u,t) + g(\bar{x}(t),t) =$$

$$= \begin{cases} (0,1) \cdot \begin{pmatrix} \bar{x}_2(t) \\ u \end{pmatrix} - 1, & if \quad t \in \left[T_{n0}, T_{n0} + \frac{1}{2^{n/2}} \right] \\ (0,1) \cdot \begin{pmatrix} \bar{x}_2(t) \\ u \end{pmatrix} + 1, & if \quad t \in \left(T_{n0} + \frac{1}{2^{n/2}}, T_{n1} \right] \end{cases} =$$

$$= \begin{cases} u - 1, \ if \quad t \in \left[T_{n0}, T_{n0} + \frac{1}{2^{n/2}} \right], \\ u + 1, \ if \quad t \in \left(T_{n0} + \frac{1}{2^{n/2}}, T_{n1} \right]. \end{cases}$$

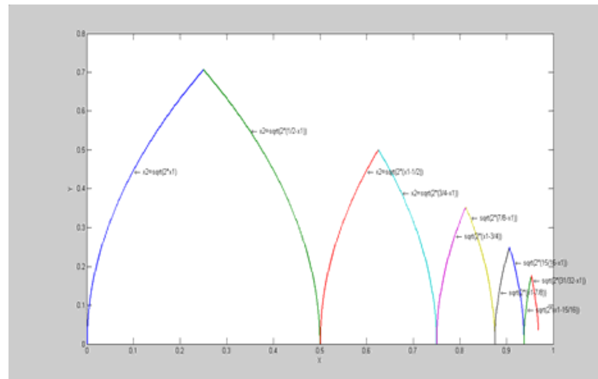

Fig. 1.

Taking into account expression for the subset $U(\bar{x}(t)), \ t \in [0, \bar{T}]$ we have

$$U(\bar{x}(t)) = \left\{ |u| \le 1 : \begin{array}{ll} u \le 1, & t \in \left[T_{n0}, T_{n0} + \frac{1}{2^{n/2}} \right], \\ u \le -1, & t \in \left(T_{n0} + \frac{1}{2^{n/2}}, T_{n1} \right]. \end{array} \right.$$

In other words, for all $T_n, \quad n = 1,2,...$ and $t \in [0, \bar{T}]$

$$U(\bar{x}(t)) = \begin{cases} u \le 1, & t \in \left[T_{n0}, T_{n0} + \frac{1}{2^{n/2}} \right], \\ \{-1\}, & t \in \left(T_{n0} + \frac{1}{2^{n/2}}, T_{n1} \right]. \end{cases}$$

Hence, optimal control function is

$$\bar{u}((t) = \begin{cases} 1, & if \quad t \in \left[T_{n0}, T_{n0} + \frac{1}{2^{n/2}} \right] \\ -1, & if \quad t \in \left(T_{n0} + \frac{1}{2^{n/2}}, T_{n1} \right] \end{cases} \qquad n = 1,2,3,...$$

Consider the auxiliary problem

$$\begin{cases}\dot{x}_1 = x_2 \\ \dot{x}_2 = u,\end{cases} \quad |u| \le 1, \quad \hat{x}_\alpha = (0,0), \quad \hat{x}_\beta = x(T),$$

$$T \to \min.$$

As a similar solution $u_0(t),\ t \in [0, \overline{T}]$ of the problem we can take $u_0(t) = 1,\ t \in [0, \overline{T}].$ Because,

$T(U,1) = (-\infty, 0]$ and

$$T(U(\overline{x}(t)), \overline{u}(t)) = \begin{cases} (-\infty, 0], & t \in \left[T_{n0}, T_{n0} + \dfrac{1}{2^{n/2}}\right], \\ \{0\}, & t \in \left(T_{n0} + \dfrac{1}{2^{n/2}}, T_{n1}\right] \end{cases}$$

$$n = 1,2,3,...$$

hence, we have

$$T(U(\overline{x}(t)), \overline{u}(t)) \subset T(U, u_0(t)), \quad a.e. \quad t \in [0, \overline{T}].$$

Thus, the condition of the existence of the similar solution is satisfied, that is there exists the nontrivial absolutely continuous adjoint function $\psi(t) = (1, \overline{T} - t),\ t \in [0, \overline{T}]$, as a solution of the adjoint system of differential equations

$$\begin{cases}\dot{\psi}_1 = 0, \\ \dot{\psi}_2 = -\psi_1,\end{cases}$$

for which the maximum condition

$$H(U(\overline{x}(t)), \psi(t)) = (\overline{u}(t), \psi(t)), \ a.e. \quad t \in [0, \overline{T}].$$

holds.

The graph of the corresponding optimal trajectory $\overline{x}(t)$ is given in the fig.1.

Conclusion

Thus, we constructed the example of time optimal control problem with active time varying state constraint, where the optimal trajectory achieves the boundary of the state constraint infinitely number times, the adjoint function is absolutely continuous, non trivial and at the same time the adjoint system is homogeneous.

Note that, the results obtained in this study include the entire regular optimal trajectory, i. e. the optimal trajectory is investigated as a whole, not dividing it to boundary or interior parts.

The advantage of this result is the fact that the adjoint equation is much simpler and has the same form as in optimal control problems without state constraints and regular trajectory in this case may be irregular for the whole set U.

A specialty of this work is also that the maximum condition is not taken on a set U, but the subset $U(\overline{x}(t)) \subset U$, as done in [14], [15].

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