# Adaptive Iterative Learning Control for a Class of Linear Time-varying Systems

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**Abstract**

With the combination of the model reference adaptive control and iterative learning control, a model reference adaptive iterative learning control algorithm was proposed for a class of first order linear time-varying systems which are BIBO stable and repeatable in a finite time interval . By means of Lyapunov technique , an iterative learning control law with adaptive update law for time-varying inertial parameter was derived . The boundedness can be guaranteed for tracking error, parameter errors and control signal , when the number of iteration trends to infinite, the tracking error will converge to zero uniformly with respect to the finite time interval.

***Keywords:*** *linear time-varying systems, Iterative learning control, Lyapunov function, Adaptive update law*

## 1. Introduction

Iterative learning control (ILC) is a scheme that learns and improves the performance of a system when the control task repeats [1]. For processes in repetitive operation mode, e.g., casting[2], rapid thermal processing[3], chemical polymerization/crystallization[4] industrial injection molding[5], ILC has achieved remarkable control performances. In those applications, different ILC algorithms, from P-type ILC to higher-order PD-type ILC, have been explored and tested. The main idea of ILC is to incorporate control and error information of the previous iterations into the control for the current iteration so as to improve the tracking accuracy, and ultimately achieve the desired control performance. ILC is playing an important role in controlling repeatable processes with parametric or non-parametric uncertainties [6]. For a class of time-varying systems which are BIBO stable and repeatable in a finite time interval , Frueh[7] proposed an adaptive iterative learning controller with parameters based on Lyapunov stability theory. Xu [ 8] proposed the ILC method to the time-varying systems based on composite energy function . The method is an adaptive iterative algorithm which is applied to the iterative field and the time-varying systems and they are repeatable in a finite time interval..

In this paper, a class of first order linear time-varying systems which are BIBO stable and repeatable in a finite time interval. The inertial parameter $a_p(t)$ and high frequency $b_p(t)$ are all time-varying. The Adaptive Iterative Learning Control is given by the corresponding model. So the method would realize not only the learn of the trajectory, but also of the system. By means of Lyapounov technique, an Iterative Learning Control law with adaptive update law for time-varying inertial parameter and high frequency gain was derived.

## 2. Problem Formulation

Considering the class of first order linear time-varying systems which are BIBO stable and repeatable in a finite time interval $[0, T]$.

$$\dot{y}^{i} = -a_p(t) y_p^i(t) + k_P(t) u^i(t) \qquad (1)$$

Where $y_p^i(t)$ and $u^i(t)$ are the output and input at time $t$ of the $i$ th iteration , respectively, $t \in \{0, 1, \cdots, T\}$ and $i = 0, 1, 2, \cdots, n$, $a_P(t), k_p(t)$ are unknown bounded time-varying parameter.

Given the desired trajectory $y_m(t)$, which is generated by the following system over $[0, T]$

$$\dot{y}_m^i = -a_m y_m^i(t) + k_m r^i(t) ,$$

where $r^i(t)$ is bounded continuous input, $a_m, k_m$ are constant which are positive. The tracking error at $i$ th cycle of Iterative Learning Control is denoted by

$$e^i(t) = y_p^i(t) - y_m^i(t) \qquad (2)$$

We assume that the system $(1)$ satisfy the initial condition: .i.e. $e^i(0) = 0, i \in Z_+$

The control target is to find a sequence of appropriate control input $u^i(t)$ so that the tracking error $e^i(t) = y_p^i(t) - y_m^i(t)$ converges to zero as the iteration number $i$ approaches infinity and to retain all the signals in the system bounded.

## 3. Adaptive Iterative Learning Control and Convergence

To realize the control target, we combined the model reference adaptive control and iterative learning control to design the iterative learning controller

$$u^i(t) = \hat{a}^i(t) y_p{}^i(t) + \hat{k}^i(t) r^i(t) \qquad (3)$$

where $\hat{a}(t)$, $\hat{b}(t)$ are $i$ th iteration, which updating law as follows:

$$\begin{aligned} \hat{k}^i(t) &= k^{i-1}(t) - \alpha_1 e^i(t) r^i(t) \\ \hat{a}^i(t) &= \hat{a}^{i-1}(t) - \alpha_2 e^i(t) y_p^i(t) \\ \hat{k}^{i+1}(0) &= \hat{k}^i(T) \end{aligned} \qquad (4)$$

where $\alpha_1$, $\alpha_2$ are positive constants and $\hat{a}_0(t)$, $\hat{k}^1(t)$ are bounded over finite interval $[0, T]$. Differentiating $(2)$ with $t$ and considering $(1)$ and $(3)$, is given below

$$\begin{aligned} \dot{e}^i(t) &= \dot{y}_p^i(t) - \dot{y}_m^i(t) \\ &= -a_m e^i(t) + (k_p(t)\hat{k}^i(t) - k_m)\, r^i(t) \\ &\quad + (a_m - a_P(t) + k_p(t)\,\hat{a}^i(t))\, y_p^i(t). \end{aligned}$$

Let $\tilde{k}^i(t) = k_p(t)\hat{k}^i(t) - k_m$, $\tilde{a}^i(t) = a_m - a_P(t) + k_p(t)\,\hat{a}^i(t)$, then

$$\begin{aligned} \dot{e}^i(t) &= \dot{y}_p^i(t) - \dot{y}_m^i(t) \\ &= -a_m e^i(t) + \tilde{k}^i(t)\, r^i(t) + \tilde{a}^i(t)\, y_p^i(t) \quad (5) \end{aligned}$$

The convergence of the algorithm above is given by the following theorem.

**Theorem 1** Considering the system given by $(1)$ and the model reference ,for a finite time interval $[0, T]$ and any $i \in Z_+$, , the input reference $r^i(t)$ is bounded and continuous . Then the initial condition $e^i(0) = 0, i \in Z_+$, the learning control law $(3)$, and the parameter updating law $(4)$, guarantee that

1) For any $i \in Z_+$, and $t \in [0,\ T]$ ,$e^i(t)$, $\tilde{k}^i(t)$ and $\tilde{a}^i(t)$ are bounded;

2) The tracking error converges to zero uniformly over the finite time interval $[0,\ T]$ as $i$ approaches to infinity;

3) The parameter estimation $\hat{k}^i(t)$ converges to $k^\infty(t)$ and $\hat{a}^i(t)$ converges to $a^\infty(t)$ uniformly over the finite time interval $[0,\ T]$ as $i$ approaches to infinity ,i.e. $\lim_{i\to\infty} \hat{k}^i(t) = k^\infty(t)$ and $\lim_{i\to\infty} \hat{a}^i(t) = a^\infty(t)$ ,

where $k^{\infty}(t)$, $a^{\infty}(t)$ are bounded time-varying parameter respectively .

4) for any $i \in Z_+$, $t \in [0,\ T]$, the control signals $u^i(t)$ are bounded。

Proof： Define the Lyapunov function as

$$W^i(t) = V(e^i(t)) + \frac{1}{2\alpha_1}\int_0^t (\tilde{k}^i(t))^2 d\tau + \frac{1}{2\alpha_2}\int_0^t (\tilde{a}^i(t))^2 d\tau, \qquad (6)$$

where $V(e^i(t)) = \frac{1}{2}(e^i(t))^2$ , To facility the statement , we omit the parameter $t$ in the function as follow .

Define the difference of Lyapunov functions of two iterative learning cycle, adjoining neighbour as $\Delta W^i(t) = W^i(t) - W^{i-1}(t),$ then

$$\Delta W^i(t) = V(e^i(t)) - V(e^{i-1}(t)) + \frac{1}{2\alpha_1}\int_0^t (\tilde{k}^i(t) - \tilde{k}^{i-1}(t))^2 d\tau + \frac{1}{2\alpha_2}\int_0^t (\tilde{a}^i(t) - \tilde{a}^{i-1}(t))^2 d\tau \qquad (7)$$

Making the parameter adaptive to updating law (4) and the same initial condition, we have

$$V(e^i(t)) = \int_0^t \dot{V}(e^i) d\tau + V(e^i(0))$$

$$= \int_0^t e^i \left( -a_m e^i(t) + \tilde{k}^i(t)\ r^i(t) + + \tilde{a}^i(t)\ y_p^i(t) \right) d\tau + V(e^i(0))$$

$$= -a_m \int_0^t (e^i)^2\ d\tau + \int_0^t \tilde{k}^i(t)\ r^i(t)\ e^i(t)\ d\tau + \int_0^t e^i(t)\ \tilde{a}^i(t)\ y_p^i(t)\ d\tau \ .$$

Based on the parameter's adaptive performance to the updating law (4) , the second and third expression an rewriting as follows respectively

$$\frac{1}{2\alpha_1}\int_0^t (\tilde{k}^i(t) - \tilde{k}^{i-1}(t))^2 d\tau$$

$$= \frac{1}{2\alpha_1}\int_0^t (\tilde{k}^i(t)^2 - (k^i + \alpha_1 e^i r^i)^2) d\tau - \int_0^t e^i(t)\ \tilde{a}^i(t)\ r^i(t)\ d\tau - \frac{\alpha_1}{2}\int_0^t (e^i)^2\ (r^i(t))^2\ d\tau\ ;$$

$$\frac{1}{2\alpha_2}\int_0^t (\tilde{a}^i(t) - \tilde{a}^{i-1}(t))^2 d\tau$$

$$= \frac{1}{2\alpha_1}\int_0^t (\tilde{a}^i(t))^2 - (a^i + \alpha_2 e^i y_p{}^i)^2 d\tau - \int_0^t e^i(t)\ \tilde{a}^i(t)\ y_p^i(t)\ d\tau - \frac{\alpha_2}{2}\int_0^t (e^i)^2\ (y_p^i(t))^2\ d\tau\ .$$

So

$$\Delta W^i(t) = -a_m \int_0^t (e^i)^2\ d\tau - \frac{\alpha_1}{2}\int_0^t (e^i)^2\ (r^i(t))^2\ d\tau - \frac{\alpha_2}{2}\int_0^t (e^i)^2\ (y_p^i(t))^2\ d\tau - \frac{1}{2}(e^{i-1})^2 \le -a_m \int_0^t (e^i)^2\ d\tau\ .$$

For any $i \in Z_+$, let $t = T$ , we have

$$\Delta W^i(T) = -a_m \int_0^T (e^i)^2\ d\tau - \frac{\alpha_1}{2}\int_0^T (e^i)^2\ (r^i(t))^2\ d\tau - \frac{\alpha_2}{2}\int_0^T (e^i)^2\ (y_p^i(t))^2\ d\tau - \frac{1}{2}(e^{i-1}(T))^2$$

$$\le -a_m \int_0^T (e^i)^2\ d\tau \le 0\ . \qquad (8)$$

Therefore $W^i(T)$ is not increasing along the iteration axis, so we can guarantee the $W^i(T)$ bounded when $W^1(T)$ is bounded。

For any $i \in Z_+$, and $t \in [0,\ T]$ ,We proove that $e^i(t)$ , $\tilde{k}^i(t)$ and $\tilde{a}^i(t)$ are bounded using mathematical induction . At first , we proof $e^1(t)$ , $\tilde{k}^1(t)$ and $\tilde{a}^1(t)$ are bounded

Since

$$W^1(t) = \frac{1}{2}(e^1)^2 + \frac{1}{2\alpha_1}\int_0^t (\tilde{k}^1(t))^2 d\tau + \frac{1}{2\alpha_2}\int_0^t (\tilde{a}^1(t))^2 d\tau \quad (9)$$

Differentiating (9) with $t$ and combining the parameter which is adaptive to updating law (4), we have

$$\dot{W}^1(t) = e^1(a_m e^1(t) + \tilde{k}^1(t)\, r^1(t) + \tilde{a}^1(t)\, y_p^1(t)) + \frac{1}{2\alpha_1}(\tilde{k}^1(t))^2 + \frac{1}{2\alpha_2}(\tilde{a}^1(t))^2$$

$$= a_m(e^1(t))^2 + \tilde{k}^1(t)\, r^1(t)\, e^1 + \tilde{a}^1(t)\, y_p^1(t)\, e^1 + \frac{1}{2\alpha_1}(k^0(t) - \alpha_1 e^1(t) r^1(t))^2 + \frac{1}{2\alpha_2}(a^0(t) - \alpha_2 e^1(t) y_p^1(t))^2$$

$$= -a_m(e^1(t))^2 + (k^0(t) - \alpha_1 e^1(t) r^1(t))\, r^1(t)\, e^1 + (a^0(t) - \alpha_2 e^1(t) y_p^1(t))\, y_p^1(t)\, e^1 + \frac{1}{2\alpha_1}(\tilde{k}^0)^2 - \tilde{k}^0 r^1(t)\, e^1 + \frac{\alpha_1}{2}(e^1(t) r^1(t))^2$$

$$= -a_m(e^1(t))^2 - \frac{\alpha_1}{2}(e^1 r^1)^2 - \frac{\alpha_2}{2}(e^1 y_p^1)^2 + \frac{1}{2\alpha_2}(\tilde{a}^0)^2 + \frac{1}{2\alpha_2}(\tilde{k}^0)^2$$

$$\le \frac{1}{2\alpha_1}(\tilde{k}^0)^2 + \frac{1}{2\alpha_2}(\tilde{a}^0)^2 \quad (10).$$

Since $\tilde{a}^0$ and $\tilde{k}^0$ are bounded and $W^1(0) = \frac{1}{2}(e^1(0))^2 = 0$, there exist a positive constant

$$M_0 = \max_{t\in[0,\ T]}[(\tilde{k}^0(t)^2/2 + \tilde{a}^0(t)^2/2]$$

so that $W^1(T) \le M_0 T + \frac{1}{2}(e^1(0))^2$. Hence $W^1(t)$ is bounded, Therefore, each term of $W^1(t)$ is bounded, so $e^1(t)$ and $\tilde{k}^1(t)$ are bounded. We also know that $y_m^i(t)$ is bounded by the assumption of continuous input $r^i(t)$ and $y_p^i(t) = y_m^i(t) + e^i(t)$, so $y_p^i(t)$, $\tilde{k}^1(t)$ and $\tilde{a}^1(t)$ are bounded. We also know that $\tilde{a}^1(t)$ is bounded by (4).

Now we assume that $e^{i-1}(t)$, $\tilde{k}^{i-1}(t)$ and $\tilde{a}^{i-1}(t)$ are bounded for any $t \in [0,\ T]$, we proof that $e^i(t)$, $\tilde{k}^i(t)$ and $\tilde{a}^i(t)$ are bounded.

We have $\hat{a}^i(t) - \hat{a}^{i-1}(t) = -\alpha_2 e^i(t) y_p^i(t)$ by (4).

Differentiating $W^i(t)$ with $t$ and combining (4), we have

$$\dot{W}^i(t) = -a_m(e^i(t))^2 - \frac{1}{2\alpha_1}(\tilde{k}^i)^2 - \frac{1}{2\alpha_2}(\tilde{a}^i)^2 - \frac{1}{2\alpha_1}(\tilde{k}^i - \tilde{k}^{i-1})^2 - \frac{1}{2\alpha_1}(\tilde{a}^i - \tilde{a}^{i-1})^2 + \frac{1}{2\alpha_2}(\tilde{a}^{i-1})^2 + \frac{1}{2\alpha_2}(\tilde{k}^{i-1})^2$$

$$\leq \frac{1}{2\alpha_1}(\tilde{k}^{i-1})^2 + \frac{1}{2\alpha_2}(\tilde{a}^{i-1})^2 \quad (11)$$

So $\dot{W}^i(t)$ is bounded for any $t \in [0,\ T]$. Considering the same initial condition $e^i(0) = 0, i \in Z_+$, the initial value of $W^i(t)$ : $W^i(0) = \frac{1}{2}(e^i(0))^2 = 0$ is bounded .

Based on the boundedness of $W^i(0)$ and $\dot{W}^i(t)$, we have $W^i(t)$ is bounded for any $i \in Z_+$ and any $t \in [0,\ T]$. So each term of $W^i(t)$ is bounded , i.e. $e^i(t)$, $\tilde{k}^i(t)$ and $\tilde{a}^i(t)$ are bounded. Therefore the conclusion (1) of Theorem 1 is proved .

(2) We prove the tracking error converges to zero uniformly over the finite time interval $[0,\ T]$ as $i$ approaches to infinity.

For $i = T$, the Lyapunov functions can be expressed by

$$W^i(T) = W^1(T) - \sum_{j=2}^{i} \Delta W^i(T)$$

By (5) We have

$$W^i(T) \leq W^1(T) - a_m \sum_{j=2}^{i} \int_0^t e^1(\tau))^2 d\tau .$$

So

$$\lim_{i\to\infty} W^i(T) + \lim_{i\to\infty} a_m \sum_{j=2}^{i} \int_0^t e^j(\tau))^2 d\tau$$

$$\leq W^1(T).$$

Since $W^1(T)$ and $W^i(T)$ are bounded $a_m \sum_{j=2}^{i} \int_0^t e^1(\tau))^2 d\tau$ is bounded , so $\lim_{i\to\infty} \int_0^T e^j(\tau))^2 d\tau = 0$ by Barbalat Lemma .

Since $e^i(t)$, $\tilde{k}^i(t)$ and $\tilde{a}^i(t)$ are bounded for any $i \in Z_+$ and any $t \in [0,\ T]$, furthermore , $r^i(t)$ and $y_p^i(t)$ are bounded , each term of (5) is bounded. Hence $\dot{e}^i(t)$ is bounded by (5) and $e^i(t)$ is equicontinuous By the literatrue [9], we have

$$\lim_{i\to\infty} e^i(t) = 0. \quad (12)$$

Therefore, the tracking error converges to zero uniformly over the finite time interval $[0,\ T]$ as $i$ approaches to infinity. The conclusion (2) of Theorem 1 is proved .

(3) We proove that $\lim_{i\to\infty} \hat{k}^i(t) = k^\infty(t)$ and $\lim_{i\to\infty} \hat{a}^i(t) = a^\infty(t)$ for any $t \in [0,\ T]$,

By (4) and (8)

$$\lim_{i\to\infty} \hat{k}^i(t) = \lim_{i\to\infty} [\hat{k}^{i-1}(t) - \alpha_1 e^i(t) r^i(t)]$$

$$= \lim_{i\to\infty} \hat{k}^{i-1}(t) \overset{\Delta}{=} k^\infty(t)$$

and

$$\lim_{i\to\infty} \hat{a}^i(t) = \lim_{i\to\infty} [\hat{a}^{i-1}(t) - \alpha_2 e^i(t) y_p^i(t)]$$

$$= \lim_{i\to\infty} \hat{a}^{i-1}(t) \overset{\Delta}{=} a^\infty(t).$$

The conclusion (3) of Theorem 1 is proved .

(4) Since $e^i(t)$, $\tilde{k}^i(t)$ and $\tilde{a}^i(t)$ are bounded by (1) of this Theorem and $r^i(t)$ and $y_p^i(t)$ are bounded, we easily know that $u^i(t)$ is bounded by (3).

This completes the proof of Theorem 1 .

## 4. Conclusion

With the combination of module reference adaptive control and ILC, a module reference adaptive ILC control algorithm was proposed for a class of first order linear time-varying systems which are BIBO stable and repeatable in a finite time interval. The approach is suitable to control the systems with rapidly time-varying parameters. The boundedness can be guaranteed for tracking error, parameter errors and control signal , when the number of iteration trends to infinite, the tracking error will converge to zero uniformly with respect to the finite time interval. The approach can be extended to the class of higher order linear time-varying systems which are BIBO stable and repeatable in a finite time interval.

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