Improved grey derivative of grey Verhulst model and its application

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Abstract

Based the principle and characteristic of grey Verhulst mode, the cause of grey Verhulst model's inaccuracy is analyzed , the new formula of grey derivative is strutted and the unbiased grey Verhulst model is given in this paper. The new modeling method improves the simulation precision and extends the application scope of grey verhulst model. Some examples are also given to show that the precision of the new model is very high.

Keywords: *grey Verhulst model, grey derivative, unbiased, application*

1. Introduction

The grey model applied in many areas since it was introduced by professor Deng. As this model needs fewer samples and the computation is simple, it has superiority than the traditional estimate method. As the grey Verhulst model modeling by the 1-AGO series, the scope of the traditional Verhulst model has been expanded to the series of approximal single peak. If the original series is an s-type form, and the size of sample is small, we can build a grey Verhulst model directly by the original series. We can see that the grey Verhulst model is superior to the traditional Verhulst model in the terms of scope, hence, the grey Verhulst model has been widely applied in recent years. Many scholars have conducted a study on grey Verhulst model, mainly on the several improvements to the model, the optimization of modeling, the optimization of grey derivative, the optimization of background value, the optimization of initial value and the optimization of calculation method of parameters, so as to further improve the accuracy and scope of models.

Based the principle and characteristic of grey Verhulst mode, the cause of grey Verhulst model's inaccuracy is analyzed , the new formula of grey derivative is strutted and the unbiased grey Verhulst model is given in this paper. The new modeling method improves the simulation precision and extends the application scope of grey verhulst model.

2. The definition of grey Verhulst model

Let $x^{(0)}=\left\{x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right\}$ be nonnegative series, $x^{(1)}=\left\{x^{(1)}(1),x^{(1)}(2),\cdots,x^{(1)}(n)\right\}$ be the 1-AGO series of $x^{(0)}$ and $x^{(1)}(k)=\sum_{i=1}^{k}x^{(0)}(i)\,,k=1\,,2\,,\cdots,n$.

$z^{(1)}=(z^{(1)}(2)\,,z^{(1)}(3)\,,\cdots,z^{(1)}(n)\,)$ be the MEAN series of $x^{(1)}$ and

$z^{(1)}(k)=0.5(x^{(1)}(k)+x^{(1)}(k-1))$, $k=2,3,\cdots,n$

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Definition 1: Let $x^{(0)}$ be nonnegative series, $x^{(1)}$ be the 1- AGO series of $x^{(0)}$, $z^{(1)}$ be the MEAN series of $x^{(1)}$, we call $x^{(0)}(k)+az^{(1)}(k)=b(z^{(1)}(k))^2$ be the grey differential equation of grey Verhulst model.

Definition 2: Let $x^{(0)}$ be nonnegative series, $x^{(1)}$ be the 1- AGO series of $x^{(0)}$, we call $\frac{dx^{(1)}}{dt}+ax^{(1)}=b(x^{(1)})^2$ be the whitened differential equation of grey Verhulst model.

Theorem 1: Let $x^{(0)}$ be nonnegative series, $x^{(1)}$ be the 1- AGO series of $x^{(0)}$, $z^{(1)}$ be the MEAN series of $x^{(1)}$ and $\alpha=(a,b)^T$ be the parameter of the equation.

Let $Y=\begin{pmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{pmatrix}$, $B=\begin{pmatrix} -z^{(1)}(2) & z^{(1)}(2)^2 \\ -z^{(1)}(3) & z^{(1)}(3)^2 \\ \vdots & \vdots \\ -z^{(1)}(n) & z^{(1)}(n)^2 \end{pmatrix}$

Then, 1) from to the method of least square, we have $\hat{\alpha}=(B^TB)^{-1}B^TY$;

2) The continuous solution of the equation is

$$\hat{x}^{(1)}(k+1)=\frac{ax^{(1)}(1)}{bx^{(1)}(1)+(a-bx^{(1)}(1))e^{ak}}$$

3) The discrete solution of the equation is $\hat{x}^{(1)}(k+1)=\frac{ax^{(1)}(1)}{bx^{(1)}(1)+(a-bx^{(1)}(1))e^{ak}}$ $k=2,3,\ldots$;

4) The original value is $\hat{x}^{(0)}(k+1)=\hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k)$ $k=1,2,\ldots$.

3. The construction of grey derivative

From theorem 1, we know that based on the whitened differential equation $\frac{dx^{(1)}}{dt}+ax^{(1)}=b(x^{(1)})^2$, professor Deng derived the grey differential equation $x^{(0)}(k)+az^{(1)}(k)=b(z^{(1)}(k))^2$ by replacing the derivative of discrete series $x^{(1)}(k)$ at point k by the difference (that is: $\left.\frac{dx^{(1)}}{dt}\right|_{t=k}=x^{(1)}(k)-x^{(1)}(k-1)=x^{(0)}(k)$) and the background value $x^{(1)}(k)$ by $z^{(1)}(k)$. However, this kind of approximate method caused much great simulation error of grey Verhulst model. Thus, now to deal with the derivative of discrete series plays a key role in improving the simulation precision of grey Verhulst model.

Based on the continuous solution of the equation $\hat{x}^{(1)}(k+1)=\frac{ax^{(1)}(1)}{bx^{(1)}(1)+(a-bx^{(1)}(1))e^{ak}}$, we get $\hat{x}^{(1)}(k+1)=\frac{1}{\frac{b}{a}+(\frac{1}{x^{(1)}(1)}-\frac{b}{a})e^{ak}}$, so we think it is reasonable that the 1-AGOseries $x^{(1)}$ is a countdown form of a homogeneous exponential series. Therefore, let $x^{(1)}(k)=\frac{1}{Be^{Ak}+C}$. It suffices to consider the discrete variable k as a continuous one and let the derivative of series $x^{(1)}$ be the grey derivative of the grey

Verhulst model.

The improved grey derivative is:

$$\frac{dx^{(1)}(k)}{dk}=\left(\frac{1}{Be^{Ak}+C}\right)'$$

$$=\frac{-BAe^{Ak}}{(Be^{Ak}+C)^2}=\frac{-A(\frac{1}{x^{(1)}(k)}-C)}{\frac{1}{x^{(1)}(k)^2}}$$

$$=-A(x^{(1)}(k)-Cx^{(1)}(k)^2)$$

$$=-Ax^{(1)}(k)(1-Cx^{(1)}(k))$$

We only need to find the value of A and C, now, establishing the following equations:

$$\begin{cases}\frac{1}{x^{(1)}(k-1)}=Be^{A(k-1)}+C & (1)\\ \frac{1}{x^{(1)}(k)}=Be^{Ak}+C & (2)\\ \frac{1}{x^{(1)}(k+1)}=Be^{A(k+1)}+C & (3)\end{cases}$$

From $\frac{(3)-(2)}{(2)-(1)}$ we have:

$$A=\ln\frac{x^{(0)}(k+1)x^{(1)}(k-1)}{x^{(0)}(k)x^{(1)}(k+1)}$$

From (2) we have: $Be^{Ak}=\frac{1}{x^{(1)}(k)}-C$, put it into (1), then: $C=\frac{x^{(1)}(k)-x^{(1)}(k-1)e^{-A}}{x^{(1)}(k-1)x^{(1)}(k)(1-e^{-A})}$,and

$$C=\frac{x^{(1)}(k)x^{(0)}(k+1)-x^{(0)}(k)x^{(1)}(k+1)}{x^{(1)}(k)(x^{(1)}(k-1)x^{(0)}(k+1)-x^{(0)}(k)x^{(1)}(k+1))}$$

So, The improved grey derivative is:

$$x^{(0)}(k)^*=\frac{dx^{(1)}(k)}{dk}$$

$$=-x^{(1)}(k)\ln\frac{x^{(0)}(k+1)x^{(1)}(k-1)}{x^{(0)}(k)x^{(1)}(k+1)}$$

$$\left(1-\frac{x^{(1)}(k)x^{(0)}(k+1)-x^{(0)}(k)x^{(1)}(k+1)}{x^{(1)}(k)(x^{(1)}(k-1)x^{(0)}(k+1)-x^{(0)}(k)x^{(1)}(k+1))}\right)$$

$$=-\frac{x^{(1)}(k-1)x^{(0)}(k+1)-x^{(0)}(k+1)x^{(1)}(k)}{x^{(1)}(k-1)x^{(0)}(k+1)-x^{(0)}(k)x^{(1)}(k+1)}x^{(1)}$$

$$(k)\ln\frac{x^{(0)}(k+1)x^{(1)}(k-1)}{x^{(0)}(k)x^{(1)}(k+1)}$$

$$=\frac{x^{(0)}(k)x^{(0)}(k+1)x^{(1)}(k)}{x^{(1)}(k-1)x^{(0)}(k+1)-x^{(0)}(k)x^{(1)}(k+1)}$$

$$\ln\frac{x^{(0)}(k+1)x^{(1)}(k-1)}{x^{(0)}(k)x^{(1)}(k+1)}$$

Theorem 2: Let $x^{(0)}$ be nonnegative series, $x^{(1)}$ be the 1- AGO series of $x^{(0)}$ and $\alpha=(a,b)^T$ be the parameter of the equation.

Let $Y=\begin{pmatrix}x^{(0)}(2)^*\\ x^{(0)}(3)^*\\ \vdots\\ x^{(0)}(n-1)^*\end{pmatrix}$,

$$B=\begin{pmatrix}-x^{(1)}(2) & x^{(1)}(2)^2\\ -x^{(1)}(3) & x^{(1)}(3)^2\\ \vdots & \vdots\\ -x^{(1)}(n-1) & x^{(1)}(n-1)^2\end{pmatrix}$$

Then, 1) from to the method of least square, we have $\hat{\alpha}=(B^TB)^{-1}B^TY$;

3) The continuous solution of the equation is

$$\hat{x}^{(1)}(k+1)=\frac{ax^{(1)}(1)}{bx^{(1)}(1)+(a-bx^{(1)}(1))e^{ak}}$$

3) The discrete solution of the equation is $\hat{x}^{(1)}(k+1)=\frac{ax^{(1)}(1)}{bx^{(1)}(1)+(a-bx^{(1)}(1))e^{ak}}$ $k=2,3,\ldots$;

5) The original value is

$$\hat{x}^{(0)}(k+1)=\hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k) \quad k=1,2,\ldots.$$

Theorem 3: The improved grey derivative of grey verhulst model is unbiased(that is :simulate s-curve is completely coincidence by this model)

Proof: let 1-AGO series be $x^{(1)}(k)=\frac{1}{De^{Ak}+C}$ k=1, 2,…, n

Then

$$x^{(0)}(k)^{*}=\frac{dx^{(1)}(k)}{dk}=-Ax^{(1)}(k)(1-Cx^{(1)}(k))$$

From Theorem 2, we know $\hat{\alpha}=(B^{T}B)^{-1}B^{T}Y$,and

$$Y=\begin{pmatrix} x^{(0)}(2)^{*} \\ x^{(0)}(3)^{*} \\ \vdots \\ x^{(0)}(n-1)^{*} \end{pmatrix},$$

$$B=\begin{pmatrix} -x^{(1)}(2) & x^{(1)}(2)^{2} \\ -x^{(1)}(3) & x^{(1)}(3)^{2} \\ \vdots & \vdots \\ -x^{(1)}(n-1) & x^{(1)}(n-1)^{2} \end{pmatrix}$$

And $B^{T}B=\begin{pmatrix} \sum_{k=2}^{n-1}x^{(1)}(k)^{2} & -\sum_{k=2}^{n-1}x^{(1)}(k)^{3} \\ -\sum_{k=2}^{n-1}x^{(1)}(k)^{3} & \sum_{k=2}^{n-1}x^{(1)}(k)^{4} \end{pmatrix}$

$$(B^{T}B)^{-1}=\frac{1}{\sum_{k=2}^{n-1}x^{(1)}(k)^{2}\sum_{k=2}^{n-1}x^{(1)}(k)^{4}-(\sum_{k=2}^{n-1}x^{(1)}(k)^{3})^{2}}\begin{pmatrix} \sum_{k=2}^{n-1}x^{(1)}(k)^{4} & \sum_{k=2}^{n-1}x^{(1)}(k)^{3} \\ \sum_{k=2}^{n-1}x^{(1)}(k)^{3} & \sum_{k=2}^{n-1}x^{(1)}(k)^{2} \end{pmatrix}$$

$$B^{T}Y=\begin{pmatrix} -\sum_{k=2}^{n-1}x^{(1)}(k)x^{(0)}(k)^{*} \\ \sum_{k=2}^{n-1}x^{(1)}(k)^{2}x^{(0)}(k)^{*} \end{pmatrix}$$

So

$$\hat{\alpha}=(B^{T}B)^{-1}B^{T}Y$$

$$=\frac{1}{\sum_{k=2}^{n-1}x^{(1)}(k)^{2}\sum_{k=2}^{n-1}x^{(1)}(k)^{4}-(\sum_{k=2}^{n-1}x^{(1)}(k)^{3})^{2}}\begin{pmatrix} \sum_{k=2}^{n-1}x^{(1)}(k)^{2}x^{(0)}(k)^{*}\sum_{k=2}^{n-1}x^{(1)}(k)^{4}-\sum_{k=2}^{n-1}x^{(1)}(k)x^{(0)}(k)^{*}\sum_{k=2}^{n-1}x^{(1)}(k)^{4} \\ \sum_{k=2}^{n-1}x^{(1)}(k)^{2}x^{(0)}(k)^{*}\sum_{k=2}^{n-1}x^{(1)}(k)^{2}-\sum_{k=2}^{n-1}x^{(1)}(k)x^{(0)}(k)^{*}\sum_{k=2}^{n-1}x^{(1)}(k)^{3} \end{pmatrix}$$

put

$x^{(0)}(k)^{*}=\frac{dx^{(1)}(k)}{dk}=-Ax^{(1)}(k)(1-Cx^{(1)}(k))$ into

the above equation, we get $\hat{\alpha}=\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} A \\ AC \end{pmatrix}$, from

theorem 2,

$$x^{(1)}(k)=\frac{1}{\frac{b}{a}+(\frac{1}{x^{(1)}(1)}-\frac{b}{a})e^{ak}}=\frac{1}{(\frac{1}{x^{(1)}(1)}-C)e^{Ak}+C}=\frac{1}{De^{Ak}+C}$$

The improved grey derivative of grey verhulst model is unbiased

4. Data simulation and application

Example 1:

$$x^{(0)}=(1\ ,\ 2.089029\ ,\ 3.66633\ ,\ 3.0117)$$ [1]

The simulation of traditional grey verhulst model is:

$$\hat{x}^{(1)}(k+1)=(0.303299+0.696701e^{-1.8513493k})^{-1}$$

The simulation of improved grey derivative of grey verhulst model is:

$$\hat{x}^{(1)}(k+1) = (0.086364 + 0.913636e^{-1.347845k})^{-1}$$

From table 1, we know: the simulation of traditional grey Verhulst model with less precision, the average of fitting accuracy is only 86%, the simulation of improved grey derivative of grey Verhulst model consistent with the original data, this model eliminates the model errors of traditional grey Verhulst, is unbiased. Cases, this optimization is reasonable, effective.

Example 2: the torpedo development cost in 1995-2003 for example

The simulation of traditional grey verhulst model is:

$$\hat{x}^{(1)}(k+1) = (0.0002225 + 0.0017936e^{-1.0110725k})^{-1}$$

The simulation of improved grey derivative of grey verhulst model is:

$$\hat{x}^{(1)}(k+1) = (0.0002235 + 0.0017927e^{-1.0617287k})^{-1}$$

Table 1: Comparison of the Simulation Precision

data	traditional grey verhulst model		improved grey verhulst model	
	Simulated value	Relative error (%)	Simulated value	Relative error (%)
1	1	0	1	0
2.089029	2.423076	15.99	2.089029	0
3.66633	3.120344	-14.89	3.66633	0
3.0117	3.268012	8.51	3.0117	0

Table 2: Comparison of the Simulation Precision

year	Cumulative costs	traditional grey verhulst model		improved grey verhulst model	
		Simulated value	Relative error (%)	Simulated value	Relative error (%)
1995	496	496	0	496	0
1996	1275	1142.7	10.37	1235.6	3.09
1997	2462	2174.2	11.69	2333.7	5.21
1998	3487	3237.2	7.16	3360.0	3.64
1999	3975	3937.8	1.94	4014.3	-0.99
2000	4230	4274.3	-1.05	4304.3	-1.76
2001	4387	4411.4	-0.56	4414.5	-0.63
2002	4497	4463.5	0.74	4454.0	0.96
2003	4584	4482.8	2.21	4467.8	2.54

From table 2, we know: the average of fitting accuracy of traditional grey Verhulst model is only 86%, and the maximum relative error has reached 11.69%. the simulation of improved grey derivative of grey Verhulst model have been enhanced, the average of fitting accuracy of it is 98% , and the maximum relative error is only 5.21%. Cases, this improved gray Verhulst model enhanced the fitting accuracy, and the optimal method have higher application value.

5. Conclusion

Based the principle and characteristic of grey Verhulst mode, the cause of grey Verhulst model's inaccuracy is analyzed, the new formula of grey derivative is structed and the unbiased grey Verhulst model is given in this paper. The new modeling method improves the simulation precision and extends the application scope of grey verhulst model.

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